

Abstract

In this paper we study decay of correlations and limit theorems for generalized baker's transformations [7, 8, 3, 20, 17]. Our examples are piecewise non-uniformly hyperbolic maps on the unit square that possess two spatially separated lines of indifferent fixed points.

We obtain sharp rates of mixing for Lipschitz functions on the unit square and limit theorems for Hölder observables on the unit square. Some of our limit theorems exhibit convergence to non-normal stable distributions for Hölder observables. We observe that stable distributions with any skewness parameter in the allowable range of $[-1, 1]$ can be obtained as a limit and derive an explicit relationship between the skewness parameter and the values of the Hölder observable along the lines of indifferent fixed points.

This paper is the first application of anisotropic Banach space methods [6, 5, 9] and operator renewal theory [19, 11] to generalized baker's transformations. Our decay of correlations results recover the results of [7]. Our results on limit theorems are new for generalized baker's transformations.

Limit Theorems for Generalized Baker's Transformations

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1 Introduction

Intermittent baker's transformations (IBTs) are invertible, non-uniformly hyperbolic, and area preserving skew products on the unit square that generalize the classical baker's transformation [7, 8, 3, 20, 17].

If a map $T: X \rightarrow X$ preserves a probability measure μ , $\psi: X \rightarrow \mathbb{R}$ is in $L^\infty(\mu)$, and $\eta: X \rightarrow \mathbb{R}$ is in $L^1(\mu)$, then we define the correlation function by

$$\text{Cor}(k; \psi, \eta, T) = \left| \int \psi \circ T^k \eta \, d\mu - \int \psi \, d\mu \int \eta \, d\mu \right|.$$

If the limit of the correlation function as k tends to infinity is zero for all $\psi \in L^\infty$ and $\eta \in L^1$, then the map is strongly mixing. If $\text{Cor}(k; \psi, \eta, T) = O\left(\frac{1}{k^\nu}\right)$ for some $\nu > 0$, then we say that the correlations decay at a polynomial rate. If the rate is independent of the choice of ψ and η in some class of functions, then we say that T displays a polynomial rate of decay of correlations for observables in that class. If the class contains functions ψ and η such that¹ $\text{Cor}(k; \psi, \eta, T) \approx \frac{1}{k^\nu}$, then we say that the rate is sharp. A limit theorem is a statement of the form: If (H) and $\int \psi \, d\mu = 0$, then

$$\frac{1}{A_n} \sum_{k=0}^{n-1} \psi \circ T^k \xrightarrow{\text{dist}} Z, \quad \text{as } n \rightarrow \infty. \quad (1.1)$$

¹We use the notation $f \approx g$ to mean that both $f = O(g)$ and $g = O(f)$. This is often also denoted by $f = \Theta(g)$, however we will not use this notation.

Where (H) is a dynamical hypothesis, A_n is a sequence of real numbers, and Z is a real valued random variable. It is well known [14] that if a map displays a summable rate of decay of correlations and mild additional hypotheses are satisfied then (1.1) is satisfied with $A_n = \sqrt{n}$ and Z a normal distribution with variance determined by the correlation function. When a map displays a rate of decay of correlations that is not summable it is possible [10] to prove that (1.1) is satisfied with a different normalizing sequence and Z a stable law, which may not be normal. In this case more delicate hypotheses are required.

In [7] the authors prove that every IBT displays a sharp polynomial rate of decay of correlations for Hölder observables via the Young tower method [21]. The Young tower method relies on analyzing an expanding factor map of the hyperbolic map in question and obtaining rates of decay of correlations for the factor map. These rates are then lifted to the full hyperbolic map via *a posteriori* arguments. Operator renewal theory [19, 11, 12] has been used to obtain sharp decay of correlation estimates and convergence to stable laws when the rate of decay of correlations is not summable. Renewal methods rely on a precise spectral decomposition of the transfer operator associated to the dynamical system in question, which in this paper is the full generalized baker's transformation rather than its factor map. The renewal method is fundamentally different from the Young tower method and this paper presents an alternative proof of the sharp rates obtained by [7].

Non-normal stable distributions possess a skewness parameter that ranges in $[-1, 1]$. In most dynamical applications limit theorems exhibit convergence to a stable distribution with skewness parameter either equal to 1 or -1 . In this paper we obtain limit theorems that exhibit convergence to stable distributions with any skewness parameter in $[-1, 1]$ and derive an explicit relationship between this parameter and properties of the IBT and the observable in question. We also obtain convergence to the normal distribution with both standard and non-standard normalizing sequences.

Additionally we will analyze the transfer operators associated to IBTs directly by introducing anisotropic Banach spaces that are adapted to the dynamics. We will obtain the spectral decomposition required to apply operator renewal theory in section 6. In section 7 we recover the sharp polynomial rates of decay of correlations for Lipschitz functions. In section 8 we obtain limit theorems for IBTs, which is a new result. See section 1.1 for statements of the theorems. The Banach spaces introduced in section 5 are modeled on the work of [6, 5, 9].

1.1 Statement of results

A function $\phi: [0, 1] \rightarrow [0, 1]$ is an intermittent cut function (ICF) if it is smooth, strictly decreasing, and there exist constants $\alpha_0, \alpha_1 > 0$, $c_0, c_1 > 0$, and a differentiable functions h_0 and h_1 defined on a neighborhood of zero with

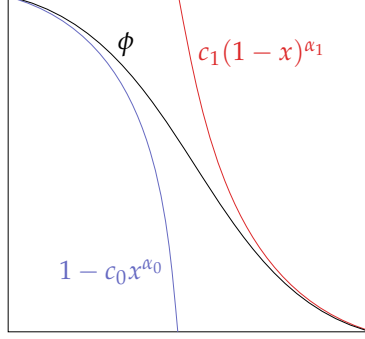


Figure 1: An intermittent cut function.

$h_j(0) = 0$ and $Dh_j(x) = o(x^{\alpha_j-1})$, such that

$$1 - \phi(x) = c_0 x^{\alpha_0} + h_0(x), \quad (1.2)$$

$$\phi(1-x) = c_1 x^{\alpha_1} + h_1(x). \quad (1.3)$$

Every IBT is uniquely determined by an ICF. We refer to the constants c_j and α_j above as the *contact coefficients* and *contact exponents* of B respectively.

Given an IBT B we will induce on a subset Λ of the unit square and apply operator renewal theory to obtain the following.

Theorem 1.1. *Suppose that $B: [0,1]^2 \rightarrow [0,1]^2$ is an Intermittent Baker's Transformation, as defined in section 2, with contact exponents $\alpha_j > 0$. Let $\alpha = \max\{\alpha_0, \alpha_1\}$. If η and ψ are Lipschitz functions on Λ , then $\text{Cor}(k; \psi, \eta, B) = O(k^{-1/\alpha})$. If additionally $\int \eta d\text{Leb} \neq 0$ and $\int \psi d\text{Leb} \neq 0$, then $\text{Cor}(k; \psi, \eta, B) \approx k^{-1/\alpha}$.*

It is important to note that we obtain a sharp decay rate in Theorem 1.1. If η and ψ are supported on Λ and $\int_{\Lambda} \eta \neq 0$ and $\int_{\Lambda} \psi \neq 0$, then eq. (7.1) shows that the rate of decay of correlation is asymptotically in bounded ratio with $n^{-1/\alpha}$.

The following is a collection of limit theorems for IBTs. See Theorem 8.6 for a precise statement.

Theorem 1.2. *Suppose that $\psi: [0,1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0,1]$ and $\int_{[0,1]^2} \psi d\text{Leb} = 0$. Let $M_0 = \int_0^1 \psi(0, y^{1+\frac{1}{\alpha_0}}) dy$ and $M_1 = \int_0^1 \psi(1, y^{1+\frac{1}{\alpha_1}}) dy$.*

- i. *If $\alpha_0, \alpha_1 < 1$, then (1.1) is satisfied with $A_n = \sqrt{n}$ and $Z = N(0, \sigma)$ where σ depends on $\text{Cor}(k; \psi, \psi, T)$ for all $k \geq 0$.*

²This hypotheses is weakened substantially in section 8.

- ii. If $\alpha_0 > \alpha_1$, $\alpha_0 > 1$, and $M_0 > 0$, then (1.1) is satisfied with $A_n = n^{\frac{\alpha_0}{\alpha_0+1}}$ and Z a stable law of index $1 + \frac{1}{\alpha_0}$, and skewness parameter 1.
- iii. If $\alpha_0 = \alpha_1 =: \alpha$, $\alpha > 1$, $M_0 > 0$ and $M_1 < 0$, then (1.1) is satisfied with $A_n = n^{\frac{\alpha}{\alpha+1}}$ and Z a stable law of index $1 + \frac{1}{\alpha}$, and skewness parameter determined by M_0 and M_1 . Any skewness parameter in $[-1, 1]$ is attainable.
- iv. If $\alpha_0 = \alpha_1 = 1$, $M_0 \neq 0$, and $M_1 \neq 0$, then (1.1) is satisfied with $A_n = \sqrt{n \log(n)}$ and $Z = N(0, \sigma^2)$ where σ^2 is determined by M_0 and M_1 .

2 Maps

Given an ICF ϕ as defined in section 1 let A denote the area below the graph of ϕ . The associated IBT B can be defined in terms of an *expanding factor map* $f: [0, 1] \circlearrowleft$ and *fibre maps* $g_x: [0, 1] \circlearrowleft$, by the formula

$$B(x, y) = (f(x), g_x(y)). \quad (2.1)$$

We define f in section 2.1 below and note that the fibre maps are defined for each $x \in [0, 1]$ by

$$g_x(y) = \begin{cases} \phi(f(x))y, & \text{if } x \in [0, A]; \\ [1 - \phi(f(x))]y + \phi(f(x)), & \text{if } x \in [A, 1]. \end{cases} \quad (2.2)$$

For convenience we introduce the following notation for iterates of B ,

$$\begin{aligned} g_x^{(0)}(y) &= y; \\ g_x^{(n+1)}(y) &= g_{f^n(x)}(g_x^{(n)}(y)), \quad n \geq 0; \end{aligned} \quad (2.3)$$

$$B^n(x, y) = (f^n(x), g_x^{(n)}(y)). \quad (2.4)$$

2.1 Expanding Factor

We define $w_0: [0, 1] \rightarrow [0, A]$ and $w_1: [0, 1] \rightarrow [A, 1]$ by

$$w_0(x) = \int_0^x \phi(t) dt, \quad (2.5)$$

$$w_1(x) = A + \int_0^x 1 - \phi(t) dt. \quad (2.6)$$

Since $\phi(0) = 1$, $\phi(1) = 0$ and ϕ is strictly decreasing we have that ϕ is strictly positive on $[0, 1)$ and hence the functions w_0 and w_1 are strictly increasing and thus are invertible. Define $f: [0, 1] \circlearrowleft$ by

$$f(x) = \begin{cases} w_0^{-1}(x), & \text{if } x \in [0, A]; \\ w_1^{-1}(x), & \text{if } x \in [A, 1]. \end{cases} \quad (2.7)$$

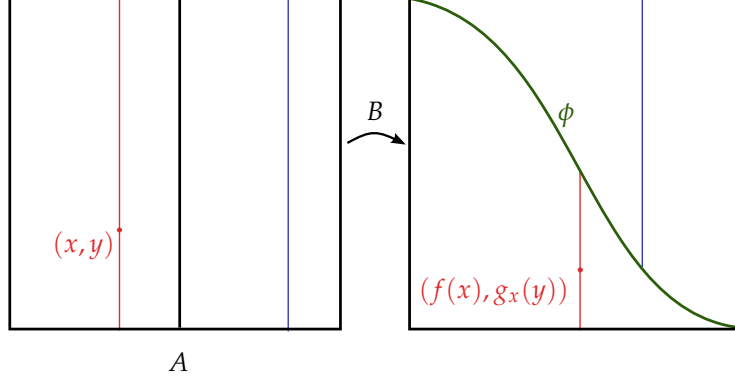


Figure 2: An intermittent bakera transformation.

Using eqs. (2.5) to (2.7) it is easy to compute

$$Df(x) = \begin{cases} [\phi(f(x))]^{-1}, & \text{if } x \in [0, A); \\ [1 - \phi(f(x))]^{-1}, & \text{if } x \in (A, 1]. \end{cases} \quad (2.8)$$

Note that $Df(x)$ approaches ∞ as x approaches A from the left or from the right. From eq. (2.7) we see that $f(0) = 0$ and $f(1) = 1$. From eq. (2.8) we see that $Df(0) = Df(1) = 1$ and therefore f has neutral fixed points at 0 and 1. It also follows from eq. (2.8) that $Df(x) \geq 1$ for all $x \neq A$, therefore f is an expanding map.

It should be noted that when the contact exponent of ϕ is α the expanding factor f is approximately $x \mapsto x(1 + cx^\alpha)$ near $x = 0$, with similar behavior near $x = 1$. From [16] Theorem 3 we might only expect a finite invariant measure for $\alpha > 1$, however f does not have bounded distortion near $x = A$ so the theorem does not apply. Note that f is the factor, by projection onto the first coordinate, of B which preserves Lebesgue measure. It follows that f must preserve Lebesgue measure. In these examples unbounded distortion near $x = A$ balances slow escape from the indifferent fixed points at $x = 0$ and $x = 1$. The map f associated to an IFC with contact exponent α preserves Lebesgue measure for any $\alpha > 0$.

3 The rate of escape from indifferent fixed points

The map f has two smooth onto branches and $Df(x) > 1$ for $x \in (0, A) \cup (A, 1)$, therefore there exist unique points $0 < p < q < 1$ such that,

$$f(p) = q, \quad f(q) = p. \quad (3.1)$$

For all $n \geq 0$ define

$$p_n = w_0^n(p), \quad q_n = w_1^n(q), \quad (3.2)$$

$$p_{n+1}^\circ = w_1(p_n), \quad q_{n+1}^\circ = w_0(q_n). \quad (3.3)$$

$$(3.4)$$

From eq. (2.7) and the definitions above it follows that for all $n \geq 0$ we have

$$f(p_{n+1}) = p_n, \quad f(q_{n+1}) = q_n, \quad (3.5)$$

$$f(p_{n+1}^\circ) = p_n, \quad f(q_{n+1}^\circ) = q_n. \quad (3.6)$$

This implies that for each $n \geq 0$ the map f carries intervals bijectively through the following orbits,

$$[p_{n+2}^\circ, p_{n+1}^\circ] \mapsto [p_{n+1}, p_n] \mapsto [p_n, p_{n-1}] \mapsto \dots \mapsto [p_1, p_0] \mapsto [p, q] \quad (3.7)$$

$$[q_{n+1}^\circ, q_{n+2}^\circ] \mapsto [q_n, q_{n+1}] \mapsto [q_{n-1}, q_n] \mapsto \dots \mapsto [q_0, q_1] \mapsto [p, q]$$

From eqs. (2.5) and (2.6) is easy to check that w_0 and w_1 have attracting fixed points at 0 and 1 respectively and that for all $n \geq 0$,

$$0 < p_{n+1} < p_n, \quad q_n < q_{n+1} < 1. \quad (3.8)$$

It follows that for all $n \geq 1$,

$$A < p_{n+1}^\circ < p_n^\circ < q, \quad p < q_n^\circ < q_{n+1}^\circ < A. \quad (3.9)$$

In the next lemma, which is Lemma 1 from [7], the asymptotics of p_n and q_n are determined for large n .

Lemma 3.1 (Lemma 1 from [7]). *If f is the expanding factor map associated to an IBT with contact exponent α and contact coefficient c , then as $n \rightarrow \infty$,*

$$p_n \sim \left(\frac{\alpha+1}{c\alpha}\right)^{1/\alpha} \left(\frac{1}{n}\right)^{1/\alpha} \quad (3.10)$$

$$1 - q_n \sim \left(\frac{\alpha+1}{c\alpha}\right)^{1/\alpha} \left(\frac{1}{n}\right)^{1/\alpha} \quad (3.11)$$

Proof. We will only prove the asymptotic for p_n the case of q_n being similar. From the definition of ϕ and w_0 we have for x sufficiently close to 0,

$$x - w_0(x) = \int_0^1 1 - \phi(t) dt = x^{\alpha+1} + o(x^{\alpha+1}).$$

Applying a Taylor expansion for $(\frac{y}{z})$ near zero and a geometric expansion we obtain

$$\frac{1}{\left(\frac{1}{y}\right)^{1/\alpha} - \left(\frac{1}{y+z}\right)^{1/\alpha}} = \frac{y^{1/\alpha}}{1 - \left(1 + \frac{y}{z}\right)^{-1/\alpha}} = \frac{y^{1/\alpha+1}}{\alpha z} + o(y^{1/\alpha}).$$

Combining the lines above we obtain for y sufficiently large

$$\frac{\left(\frac{1}{y}\right)^{1/\alpha} - w_0 \left(\left(\frac{1}{y}\right)^{1/\alpha}\right)}{\left(\frac{1}{y}\right)^{1/\alpha} - \left(\frac{1}{y+z}\right)^{1/\alpha}} = \frac{\alpha c}{\alpha + 1} \frac{1}{z} + o(1)$$

Setting $z = \frac{\alpha c}{\alpha + 1}$ and $\left(\frac{1}{y}\right)^{1/\alpha} = p_k$ for k sufficiently large we obtain

$$\frac{p_k - p_{k+1}}{p_k - \left(\frac{1}{y+z}\right)^{1/\alpha}} = 1 + o(1).$$

An induction argument shows that for all $j \geq 1$

$$\frac{p_k - p_{k+j}}{p_k - \left(\frac{1}{y+jz}\right)^{1/\alpha}} = 1 + o(1).$$

Rearranging yields

$$p_{k+j} = \left(\frac{1}{y+jz}\right)^{1/\alpha} (1 + o(1)) = z^{-1/\alpha} \left(\frac{1}{k+j}\right)^{1/\alpha} (1 + o(1)) \left(\frac{\frac{k}{j}+1}{\frac{y}{jz}+1}\right)^{1/\alpha}$$

Letting $s = \frac{1}{j}$, Taylor expanding the last term about $s = 0$, recalling that $z = \frac{\alpha c}{\alpha + 1}$, and setting $n = k + j$ we obtain,

$$\begin{aligned} p_n &= z^{-1/\alpha} \left(\frac{1}{k+j}\right)^{1/\alpha} (1 + o(1)) \left(1 + o\left(\frac{1}{j}\right)\right) \\ &= \left(\frac{\alpha+1}{\alpha c}\right)^{1/\alpha} \left(\frac{1}{n}\right)^{1/\alpha} + o\left(\left(\frac{1}{n}\right)^{1/\alpha}\right). \end{aligned}$$

The other estimate is similar. □

Lemma 3.2. *If f is the expanding factor map associated to an IBT with contact exponent α and contact coefficient c , then as $n \rightarrow \infty$,*

$$p_n - p_{n+1} \sim \frac{c}{\alpha+1} \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{1+\frac{1}{\alpha}} \quad (3.12)$$

$$q_{n+1} - q_n \sim \frac{c}{\alpha+1} \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{1+\frac{1}{\alpha}} \quad (3.13)$$

Proof. Note that by section 3 and eqs. (1.2) and (2.5) we have

$$\begin{aligned}
p_n - p_{n+1} &= p_n - w_0(p_n) \\
&= p_n - \int_0^{p_n} \phi(t) dt \\
&= \int_0^{p_n} 1 - \phi(t) dt \\
&= \int_0^{p_n} ct^\alpha + h(t) dt \\
&\sim \left(\frac{c}{\alpha+1}\right) p_n^{\alpha+1} \\
&\sim \left(\frac{c}{\alpha+1}\right) \left(\frac{\alpha+1}{c\alpha}\right)^{1/\alpha} \left(\frac{1}{n}\right)^{1/\alpha}.
\end{aligned}$$

The other estimate is similar. \square

Lemma 3.3. *If f is the expanding factor map associated to an IBT with contact exponent α and contact coefficient c , then for all n sufficiently large,*

$$\begin{aligned}
p_n^\circ - p_{n+1}^\circ &\sim \frac{c}{\alpha} \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{2+\frac{1}{\alpha}} \\
q_{n+1}^\circ - q_n^\circ &\sim \frac{c}{\alpha} \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{2+\frac{1}{\alpha}}
\end{aligned} \tag{3.14}$$

Proof. By section 3 and eqs. (1.2) and (2.6) we have

$$\begin{aligned}
p_n^\circ - p_{n+1}^\circ &= w_1(p_{n-1}) - w_1(p_n) \\
&= \int_{p_n}^{p_{n-1}} 1 - \phi(t) dt \\
&= \int_{p_n}^{p_{n-1}} ct^\alpha + h(t) dt \\
&= \left(\frac{c}{\alpha+1}\right) (p_{n-1}^{\alpha+1} - p_n^{\alpha+1}) + o\left(\int_{p_n}^{p_{n-1}} t^\alpha dt\right)
\end{aligned}$$

By eq. (3.10) and an easy Taylor expansion,

$$\begin{aligned}
p_{n-1}^{\alpha+1} - p_n^{\alpha+1} &= \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\left(\frac{1}{n-1}\right)^{1+\frac{1}{\alpha}} - \left(\frac{1}{n}\right)^{1+\frac{1}{\alpha}} \right) \\
&= \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{1+\frac{1}{\alpha}} \left(\left(1 - \frac{1}{n}\right)^{-1-\frac{1}{\alpha}} - 1 \right) \\
&= \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{1+\frac{1}{\alpha}} \left(\frac{\alpha+1}{\alpha} \frac{1}{n} + o\left(\frac{1}{n}\right) \right) \\
&= \frac{\alpha+1}{\alpha} \left(\frac{\alpha+1}{c\alpha}\right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n}\right)^{2+\frac{1}{\alpha}} + o\left(\left(\frac{1}{n}\right)^{2+\frac{1}{\alpha}}\right)
\end{aligned}$$

We conclude that as $n \rightarrow \infty$,

$$p_{n+1}^\circ - p_n^\circ = \frac{c}{\alpha} \left(\frac{\alpha+1}{c\alpha} \right)^{1+\frac{1}{\alpha}} \left(\frac{1}{n} \right)^{2+\frac{1}{\alpha}} + o \left(\left(\frac{1}{n} \right)^{2+\frac{1}{\alpha}} \right).$$

□

Lemma 3.4. *If f is the expanding factor map associated to an IBT with contact exponent α and contact constant c , then for all n sufficiently large,*

$$\begin{aligned} p_n^\circ - A &\sim \frac{1}{\alpha} \left(\frac{\alpha+1}{c\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{1}{n} \right)^{1+\frac{1}{\alpha}} \\ A - q_n^\circ &\sim \frac{1}{\alpha} \left(\frac{\alpha+1}{c\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{1}{n} \right)^{1+\frac{1}{\alpha}} \end{aligned} \quad (3.15)$$

Proof. By section 3 and eqs. (1.2) and (2.6) we have

$$\begin{aligned} p_n^\circ - A &= w_1(p_{n-1}) - w_1(0) \\ &= \int_0^{p_{n-1}} 1 - \phi(t) dt \\ &= \int_0^{p_{n-1}} ct^\alpha + h(t) dt \\ &= \left(\frac{c}{\alpha+1} \right) (p_{n-1}^{\alpha+1}) + o \left(\int_0^{p_{n-1}} t^\alpha dt \right) \\ &= \frac{1}{\alpha} \left(\frac{\alpha+1}{c\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{1}{n} \right)^{1+\frac{1}{\alpha}} + o \left(\left(\frac{1}{n} \right)^{1+\frac{1}{\alpha}} \right) \end{aligned}$$

□

Lemma 3.5. *Suppose $n \geq 0$ and that $(x, y) \in \Lambda$ is a point such that $x \in [A, q]$ and $r(x, y) = n + 2$. For all $1 \leq k \leq n + 1$, let $(x_k, y_k) = B^k(x, y)$. As $n - k \rightarrow \infty$,*

$$x_k \sim \left(\frac{\alpha+1}{c\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{1}{n-k+2} \right)^{\frac{1}{\alpha}}, \quad (3.16)$$

$$y_k \sim \left(1 - \frac{k+1}{n} \right)^{1+\frac{1}{\alpha}}. \quad (3.17)$$

Proof. By eq. (3.7), $x_k \in [p_{n-k+2}, p_{n-k+1}]$. By eq. (3.10),

$$p_{n-k+2} \sim \left(\frac{\alpha+1}{c\alpha} \right)^{1/\alpha} \left(\frac{1}{n-k+2} \right)^{1/\alpha}.$$

By eq. (3.12),

$$x_k - p_{n-k+2} \leq p_{n-k+1} - p_{n-k+2} = o \left(\frac{1}{n-k} \right)^{\frac{1}{\alpha}}.$$

This verifies the claimed asymptotic behavior of x_k .

Recall eqs. (2.2) and (2.3), and note that for $k \geq 2$

$$y_k = [\phi(x_1) + (1 - \phi(x_1))y] \prod_{j=2}^k \phi(x_j)$$

and y_1 can be obtained by omitting the product in the equation above. Applying eq. (1.2) and expanding $\log(1 - t)$ about $t = 0$, we see that as $x_j \rightarrow 0$

$$\log(\phi(x_j)) = \log(1 - cx_j^\alpha + h(x_j)) \sim -cx_j^\alpha.$$

Applying the asymptotic for x_k from above we obtain,

$$\log(\phi(x_j)) \sim -\left(\frac{\alpha+1}{\alpha}\right) \left(\frac{1}{n-j+2}\right).$$

It follows that

$$\sum_{j=2}^k \log(\phi(x_j)) \sim -\left(\frac{\alpha+1}{\alpha}\right) \sum_{j=2}^k \frac{1}{n-j+2} \sim \frac{\alpha+1}{\alpha} \log\left(\frac{n-k+1}{n}\right).$$

Therefore,

$$\prod_{j=2}^k \phi(x_j) \sim \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\alpha}}.$$

Noting that $\phi(x_1) = 1 + o\left(\frac{1}{n}\right)$ we see that,

$$y_k \sim \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\alpha}},$$

as desired. \square

4 Induced Map

In this section we will take a Intermittent Baker's Transformation that is non-uniformly hyperbolic and has unbounded distortion and construct an induced map that will enjoy uniform hyperbolicity and bounded distortion.

Given an Intermittent Baker's Transformation $B: [0,1]^2 \curvearrowright$ as defined in section 2, let $\{p, q\}$ denote the period-2 orbit of the associated factor map f that was described in section 3. Define

$$\Lambda = [p, q] \times [0, 1]. \quad (4.1)$$

We will refer to Λ as the *base* and consider first returns to Λ . Define the *return time function* $r: \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$r(x, y) = \inf \{n \in \mathbb{N} \cup \{\infty\} : B^n(x, y) \in \Lambda\}. \quad (4.2)$$

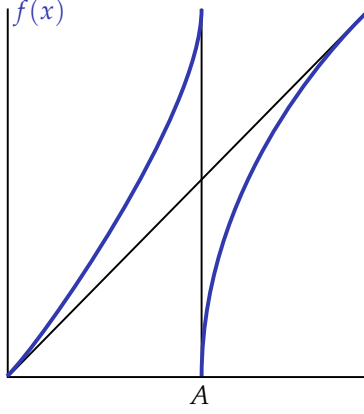


Figure 3: The expanding factor of an IBT.

The induced map $T: \Lambda \curvearrowright$, defined by

$$T(x, y) = B^{r(x, y)}(x, y), \quad (4.3)$$

maps a point in Λ to the first point along its B -orbit that lands in Λ . Let λ denote the conditional measure defined by

$$\lambda(E) = \frac{\text{Leb}(E \cap \Lambda)}{\text{Leb}(\Lambda)}. \quad (4.4)$$

Define the projection of this measure on to $[p, q]$ by

$$\mu(E) = \lambda(E \times [0, 1]). \quad (4.5)$$

Given a point (x, y) the first coordinate of a $B^n(x, y)$ is independent of y for all $n \geq 0$, similarly membership of (x, y) in Λ does not depend on y . We conclude that $r(x, y)$ does not depend on y . It follows that

$$T(x, y) = B^{r(x)}(x, y) = \left(f^{r(x)}(x), g_x^{(r(x))}(y) \right). \quad (4.6)$$

We see that T is a skew product and define a factor map $u: [p, q] \curvearrowright$ and fibre maps $v_x: [0, 1] \curvearrowright$ for each $x \in [p, q]$ by,

$$u(x) = f^{r(x)}(x), \quad (4.7)$$

$$v_x(y) = g_x^{(r(x))}(y). \quad (4.8)$$

When we refer to iterates of T we will use the notation $v_x^{(n)}$ defined analogously to eq. (2.3).

Note that by eq. (3.7) we have, for each $n \geq 0$,

$$[r = n + 2] = ((q_{n+1}^\circ, q_{n+2}^\circ] \cup [p_{n+2}^\circ, p_{n+1}^\circ)) \times [0, 1]. \quad (4.9)$$

It follows from Lemma 3.1 that,

$$\lambda[r = n] \approx \left(\frac{1}{n}\right)^{1/\alpha+2}. \quad (4.10)$$

In what follows it will be convenient to define the k -th return time $r^{(k)} : \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ by,

$$\begin{aligned} r^{(1)}(x, y) &= r(x, y) \\ r^{(k+1)}(x, y) &= r^{(k)}(x, y) + r\left(T^k(x, y)\right). \end{aligned} \quad (4.11)$$

Note that if $n = r^{(k)}(x, y)$, then n is the smallest positive integer so that the set $\{B^j(x, y) : j = 1, \dots, n\}$ contains k points in Λ .

4.1 Derivative Bounds

While an IBT is non-uniformly hyperbolic, the induced map introduced in the last section enjoys uniform hyperbolicity. For our purposes it suffices to show that the factor map u of the induced map T is a well behaved interval map meaning that it enjoys uniform expansion and bounded distortion. The following lemmas from [7] provide the necessary bounds.

Lemma 4.1 (Lemma 2 from [7]). *If*

$$\beta = \sup_{t \in [p, q]} \max\{\phi(t), 1 - \phi(t)\}, \quad (4.12)$$

then

$$\left\| [Du]^{-1} \right\|_{\sup} \leq \beta. \quad (4.13)$$

Proof. See [7]. □

Lemma 4.2 (Lemma 3 from [7]). *There exists $\kappa < \infty$ such that*

$$\left\| \frac{D^2 u}{[Du]^2} \right\|_{\sup} \leq \kappa. \quad (4.14)$$

Proof. See [7]. □

4.2 Dynamical Partitions

Our anisotropic Banach spaces will be built with respect to stable and unstable curves for the IBT. Since T is a skew product, it is easy to check that vertical lines form an equivariant family of stable curves for T . For convenience we introduce notation. For every $x \in [p, q]$, define

$$\ell(x) = \{x\} \times [0, 1]. \quad (4.15)$$

With this notation equivariance takes the form

$$T(\ell(x)) \subset \ell(u(x)). \quad (4.16)$$

It is routine to check that for every $x \in [p, q]$ the map $v_x: \ell(x) \rightarrow \ell(u(x))$ is an affine contraction by at least β .

The next lemma characterizes unstable curves for T .

Lemma 4.3. *There is an equivariant family Γ of unstable curves for T such that, each curve is the graph of a function in $C^1([p, q], [0, 1])$, the family is bounded in the C^1 norm, and the family forms a partition of Λ .*

Proof. See [18] Chapter 12. □

We define $\gamma: \Lambda \rightarrow \Gamma$ by,

$$\gamma(x, y) \in \Gamma \text{ such that } x \in \gamma(x, y). \quad (4.17)$$

Since Γ is a partition $\gamma(x, y)$ is uniquely defined.

Note that by eq. (4.9) the collection $\{[r = n] : n \geq 1\}$ is a partition mod λ of Λ , as is $\{(p, A) \times [0, 1], (A, q) \times [0, 1]\}$. For all $n \geq 1$ we define,

$$\begin{aligned} \Omega_1 &= \{[r = n] : n \geq 1\} \vee \{(p, A) \times [0, 1], (A, q) \times [0, 1]\}, \\ \Omega_{n+1} &= \Omega_1 \vee T^{-1}\Omega_n. \end{aligned} \quad (4.18)$$

All of these collections are partitions mod λ since T is measure preserving. Every cell of Ω_n is a column of the form $[a, b] \times [0, 1)$ or $(a, b] \times [0, 1]$. We define $\omega_n: \Lambda \rightarrow \Omega_n$ by,

$$\omega_n(x, y) \in \Omega_n \text{ such that } x \in \omega_n(x, y). \quad (4.19)$$

Since Ω is a partition mod λ , we have that $\omega(x, y)$ is uniquely defined for λ -a.e. (x, y) . Note that $r^{(k)}$ is measurable with respect to Ω_k .

Lastly we define measurable partitions Θ_n and maps $\theta_n: \Lambda \rightarrow \Theta_n$ by

$$\Theta_n = T^n \Omega_n \quad (4.20)$$

$$\theta_n(x, y) \in \Theta_n \text{ such that } x \in \omega_n(x, y). \quad (4.21)$$

The cells of Θ_n are strips that are bounded above and below by curves in Γ .

5 Adapted Banach Spaces

In this section we will define anisotropic Banach spaces adapted to the dynamics of the induced map T . We will begin by defining a symbolic metric on vertical lines and spaces of functions that are Hölder along each vertical line with respect to this symbolic metric.

5.1 Symbolic Metric on Stable Leaves

Define the *stable separation time* $s: \Lambda \times \Lambda \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$s((x, y), (w, z)) = \sup \{n \in \mathbb{N} : \Theta_n(x, y) = \Theta_n(w, z)\}. \quad (5.1)$$

Note that

$$s(T^k(x, y), T^k(w, z)) = s((x, y), (w, z)) + k. \quad (5.2)$$

Define the stable pseudometric $d: \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$d((x, y), (w, z)) = \beta^{s((x, y), (w, z))}. \quad (5.3)$$

where we follow the convention that $\beta^\infty = 0$. For each vertical line $\ell(x) \subset \Lambda$, let d_x denote the restriction of d to $\ell(x)$ defined for $y, z \in \ell(x)$ by,

$$d_x(y, z) = d((x, y), (x, z)). \quad (5.4)$$

It follows immediately from eq. (5.2) that,

$$d(T^k(x, y), T^k(w, z)) = \beta^k d((x, y), (w, z)). \quad (5.5)$$

5.2 Stable Hölder Spaces

Given a point $x \in [p, q]$, a bounded measurable function $h: \ell(x) \rightarrow \mathbb{C}$, and $a \in (0, 1]$, define,

$$H_x^a(h) = \sup_{y \neq z} \frac{|h(y) - h(z)|}{d_x(y, z)^a}, \quad (5.6)$$

and

$$\|h\|_x^a = \|h\|_{\sup} + H_x^a(h). \quad (5.7)$$

Let $\mathbf{H}_x^a = \{h : H_x^a(h) < \infty\}$, which is the space of a -Hölder functions on $\ell(x)$ with respect to the metric d_x .

If $\psi: \Lambda \rightarrow \mathbb{C}$ is a bounded measurable function then we define $H_x^a(\psi) = H_x^a(\psi(x, \cdot))$ and $\|\psi\|_x^a = \|\psi(x, \cdot)\|_x^a$. Fix $a \in (0, 1)$ and define

$$\|\psi\|_{\mathbf{A}} = \int_{[p, q]} \|\psi\|_x^1 d\mu(x), \quad (5.8)$$

$$\|\psi\|_{\mathbf{B}} = \int_{[p, q]} \|\psi\|_x^a d\mu(x). \quad (5.9)$$

Let \mathbf{A} denote the space of bounded measurable functions ψ with $\|\psi\|_{\mathbf{A}} < \infty$, and define \mathbf{B} similarly with respect to $\|\cdot\|_{\mathbf{B}}$. Note that $\|\cdot\|_{\mathbf{B}} \leq \|\cdot\|_{\mathbf{A}}$ and thus $\mathbf{A} \subset \mathbf{B}$.

The following observations will be useful in the proof of the Lasota-Yorke inequality in section 6.2. It follows from eq. (5.5) that for all $k \geq 0$

$$H_x^a(\psi \circ T^k) = \sup_{y \neq z} \frac{|\psi \circ T^k(x, y) - \psi \circ T^k(x, z)|}{d(T^k(x, y), T^k(x, z))^a} \quad (5.10)$$

$$\leq (\beta^a)^k H_{u^k(x)}^a(\psi). \quad (5.11)$$

If for every $x \in [p, q]$ there exists $y \in [0, 1]$ such that $\phi(x, y) = 0$, then

$$|\psi|_x^a \leq 2H_x^a(\psi).$$

Given $k \geq 0$ and $\psi \in \mathbf{B}$, define $\psi_0(x, y) = \psi \circ T^k(x, 0)$, then

$$\begin{aligned} \|\psi \circ T^k - \psi_0\|_x^a &\leq 2H_x^a(\psi \circ T^k) \\ &\leq 2(\beta^a)^k H_{u^k(x)}^a(\psi) \\ &\leq 2(\beta^a)^k \|\psi \circ T^k\|_x^a. \end{aligned} \quad (5.12)$$

5.3 Unstable Expectation Operators

For each $k \geq 1$ and bounded measurable $\psi: \Lambda \rightarrow \mathbb{C}$, define the k -th unstable average of ψ by

$$E_k \psi(x, y) = \frac{\int_{\theta_k(x, y)} \psi d\lambda}{\lambda(\theta_k(x, y))}. \quad (5.13)$$

Lemma 5.1. *For all $\psi \in \mathbf{B}$, the sequence $(E_k \psi)_{k=1}^\infty$ is Cauchy with respect to the uniform norm.*

Given $\psi \in \mathbf{B}$ define

$$E_\Gamma \psi(x, y) = \lim_{k \rightarrow \infty} E_k \psi(x, y) \quad (5.14)$$

Lemma 5.2. *For all ψ in \mathbf{B} , the function $E_\Gamma \psi$ is also in \mathbf{B} . The operator norm of $E_\Gamma: \mathbf{B} \rightarrow \mathbf{B}$ is bounded above by $[2\kappa + 1]^2$.*

Let \mathcal{B}^u denote the σ -algebra of Borel sets that are saturated³ with respect to Γ .

Lemma 5.3. *For all bounded measurable $\psi: \Lambda \rightarrow \mathbb{C}$ and $A \in \mathcal{B}^u$,*

$$\int_A E_\Gamma \psi d\lambda = \int_A \psi d\lambda. \quad (5.15)$$

³A set $E \subseteq \Lambda$ is saturated with respect to Γ if for every $(x, y) \in E$, $\gamma(x, y) \subseteq E$.

5.4 Sampling Operators

In order to define our norms we will introduce the following linear operators that sample the values of a function $\eta: \Lambda \rightarrow \mathbb{C}$ along a vertical line and produce a function that is constant along unstable curves. For each $t \in [p, q]$ define $S(t)$ acting on bounded measurable functions by

$$[S(t)\eta](x, y) = \eta(\ell(t) \cap \gamma(x, y)). \quad (5.16)$$

The commutation relation eq. (5.18) will be useful when we prove the Lasota-York inequality in section 6.2.

Given a point (x, y) in Λ , $t \in [p, q]$, and $k \geq 0$ there exists a unique point $t_k = t_k(x, y)$ such that

$$T^{-k}(\gamma(x, y) \cap \ell(t)) = \gamma(T^{-k}(x, y)) \cap \ell(t_k) \quad (5.17)$$

With eq. (5.17) we can state the commutation relation

$$\begin{aligned} [S(t)T_*^k\eta](x, y) &= \eta(T^{-k}(\gamma(x, y) \cap \ell(t))) = \eta(\gamma(T^{-k}(x, y)) \cap \ell(t_k)) \\ &= [T_*^k S(t_k)\eta](x, y). \end{aligned} \quad (5.18)$$

It is important to note that if $(w, z) \in \gamma(x, y)$, then $t_k(w, z) = t_k(x, y)$, and that for a fixed (x, y) the mapping $t \mapsto t_k(x, y)$ is the inverse of a single branch of u^k . It follows that for $(x, y) \in \Lambda$, t and s in $[p, q]$, and $k \geq 0$,

$$|t_k(x, y) - s_k(x, y)| \leq \beta^k |t - s| \quad (5.19)$$

5.5 Norms

In this section we define norms and Banach spaces adapted to the dynamics of the induced map T .

Given a bounded measurable function $\eta: \Lambda \rightarrow \mathbb{C}$ define

$$Lip_u(\eta) = \sup \left\{ \frac{|S(t)\eta - S(s)\eta|(x, y)}{|t - s|} : t, s \in [p, q], (x, y) \in \Lambda \right\}, \quad (5.20)$$

$$\|\eta\|_{\mathbf{L}} = \|\eta\|_{\sup} + Lip_u(\eta). \quad (5.21)$$

Let \mathbf{L} denote the space of bounded measurable functions η with $\|\eta\|_{\mathbf{L}} < \infty$.

For all bounded measurable functions $\eta: \Lambda \rightarrow \mathbb{R}$ define

$$\|\eta\|_{\mathcal{W}} = \sup \left\{ \int_{\Lambda} S(t)\eta \psi d\lambda : t \in [p, q], \psi \in \mathbf{A}, \|\psi\|_{\mathbf{A}} \leq 1 \right\}, \quad (5.22)$$

$$\|\eta\|_s = \sup \left\{ \int_{\Lambda} S(t)\eta \psi d\lambda : t \in [p, q], \psi \in \mathbf{B}, \|\psi\|_{\mathbf{B}} \leq 1 \right\}, \quad (5.23)$$

$$Lip_s(\eta) = \sup \left\{ \frac{\int_{\Lambda} (S(t) - S(s)) \eta \psi d\lambda}{|t - s|} : t, s \in [p, q], \psi \in \mathbf{B}, \|\psi\|_{\mathbf{B}} \leq 1 \right\}, \quad (5.24)$$

$$\|\eta\|_{\mathcal{S}} = \|\eta\|_s + Lip_s(\eta). \quad (5.25)$$

Since $\mathbf{A} \subset \mathbf{B}$ we have $\|\cdot\|_{\mathcal{W}} \leq \|\cdot\|_s \leq \|\cdot\|_{\mathcal{S}}$. Both $\|\cdot\|_{\mathcal{S}}$ and $\|\cdot\|_{\mathcal{W}}$ are bounded semi-norms on \mathbf{L} , by taking quotients $\|\cdot\|_{\mathcal{S}}$ and $\|\cdot\|_{\mathcal{W}}$ induce norms on quotient spaces of \mathbf{L} , completing these quotient spaces with respect to their norms produces Banach spaces \mathcal{S} and \mathcal{W} .

5.6 Compact Embedding

In this section we address the compact embedding hypothesis of Hennion's Theorem [13], which we will use to deduce quasi-compactness of certain renewal operators in section 6.

Lemma 5.4. *The inclusion of \mathcal{S} into \mathcal{W} is a compact embedding.*

Proof. The format of this proof is standard and can be seen for example in [15] The key observations are:

1. For each $t \in [p, q]$ the function $S(t)\eta$ is measurable with respect to the unstable σ -algebra so,

$$\int_{\Lambda} S(t)\eta \psi d\lambda = \int_{\Lambda} E_{\Gamma} [S(t)\eta \psi] d\lambda = \int_{\Lambda} S(t)\eta E_{\Gamma} \psi d\lambda.$$

2. By Lemma 5.2 E_{Γ} is bounded on \mathbf{A} .
3. Restriction to $\ell(x)$ is an isometric isomorphism of $E_{\Gamma}\mathbf{A}$ onto \mathbf{H}_x^1 for every $x \in [p, q]$.
4. For each $x \in [p, q]$ the space \mathbf{H}_x^1 is compactly embedded into \mathbf{H}_x^a . It follows that the image of the unit ball of \mathbf{A} under E_{Γ} is totally bounded in \mathbf{B} .
5. Given $\epsilon > 0$ we can select a finite set $A_{\epsilon} \subset E_{\Gamma}\mathbf{A}$ that is ϵ -dense with respect to $\|\cdot\|_{\mathbf{B}}$, and a finite set $B_{\epsilon} \subset [p, q]$ that is ϵ -dense.

6. Fix a finite ϵ -dense subset $E_\epsilon \subset [p, q]$. For any $\eta \in \mathbf{L}$ with $\|\eta\|_{\mathcal{S}} \leq 1$, $t \in [p, q]$, and $\psi \in \mathbf{A}$, we can select $\xi \in A_\epsilon$ so that $\|E_\Gamma \psi - \xi\|_{\mathbf{B}} < \epsilon$ and $s \in E_\epsilon$ such that $|t - s| < \epsilon$. We compute

$$\begin{aligned}
\left| \int_{\Lambda} S(t) \eta \psi d\lambda - \int_{\Lambda} S(s) \eta \xi d\lambda \right| &\leq \left| \int_{\Lambda} [S(s) - S(t)] \eta \psi d\lambda \right| \\
&\quad + \left| \int_{\Lambda} S(s) \eta [\psi - \xi] d\lambda \right| \\
&= \left| \int_{\Lambda} [S(s) - S(t)] \eta \psi d\lambda \right| \\
&\quad + \left| \int_{\Lambda} S(s) \eta [E_\Gamma \psi - \xi] d\lambda \right| \\
&\leq \text{Lip}_s(\eta) |t - s| \\
&\quad + \|\eta\|_{\mathcal{S}} \|E_\Gamma \psi - \xi\|_{\mathbf{B}} \\
&\leq \epsilon \|\eta\|_{\mathcal{S}}
\end{aligned}$$

From the forgoing observations it follows that $\{\eta \in \mathbf{L} : \|\eta\|_{\mathcal{S}} \leq 1\}$ is precompact in \mathcal{W} . Since \mathbf{L} is dense in \mathcal{S} we conclude that the unit ball of \mathcal{S} is precompact in \mathcal{W} . \square

6 Operator Renewal Theory

In this section we apply operator renewal theory as described in [11] to connect spectral properties of the transfer operator of the induced map T and the rate of decay of correlation for the IBT B . The following operators are the central objects of the operator renewal method.

Heuristically, if η is supported on Λ and $\int_{\Lambda} \eta \neq 0$, then the push forward distributions $B_*^n \eta$ must equilibrate to a multiple $\mathbf{1}_{[0,1]^2}$ which is the density for the preserved measure. The transfer operator B_* sends all of the mass represented by η outside of Λ . In order for $B_*^n \eta$ to attain its limiting value of $\int_{[0,1]^2} \eta d\text{Leb}$ inside of Λ mass must return to Λ . The amount of mass that has failed to return after n steps of the dynamics is $\text{Leb}[r > n]$, which provides a rough estimate for how quickly the convergence $B_*^n \eta \rightarrow \mathbf{1}_{[0,1]^2} \int_{[0,1]^2} \eta d\text{Leb}$ can occur. Theorem 1.1 shows that this rough estimate is actually sharp.

For each $n \geq 1$ and $k \geq 1$ we define operators by

$$R_n^{(k)} \eta = T_*^k \left(\mathbf{1}_{\{r^{(k)}=n\}} \eta \right), \quad (6.1)$$

$$B_n \eta = \mathbf{1}_{\Lambda} B_*^n (\mathbf{1}_{\Lambda} \eta). \quad (6.2)$$

We will always abbreviate $R_n^{(1)}$ as R_n . The operators R_k are a decomposition

of T_* by first return time. The operators B_n can be viewed as a restriction of B_*^n to an action on functions supported on Λ_* .

6.1 Renewal Equation

A key technical observation in operator renewal theory is that the generating functions defined by eqs. (6.3) and (6.4) are related by eq. (6.5).

$$B(z) = I + \sum_{n=1}^{\infty} z^n B_n \quad (6.3)$$

$$R(z) = \sum_{n=1}^{\infty} z^n R_n \quad (6.4)$$

$$B(z) = [I - R(z)]^{-1} \quad (6.5)$$

We record this fact as the following lemma.

Lemma 6.1. *For every z in the unit disk of \mathbb{C} , the operators $B(z)$ and $R(z)$ satisfy eq. (6.5).*

Proof. See [19] Proposition 1. □

In the next section we will make use of the following identities, which are routine to check,

$$R(1) = T_* \quad (6.6)$$

$$R(z)^k = \sum_{n=1}^{\infty} R_n^{(k)} z^n. \quad (6.7)$$

6.2 A Uniform Lasota-Yorke Inequality

In this section we show that $R(z)$ satisfies a uniform Lasota-Yorke inequality for $|z| \leq 1$. We also collect Bounds on the $R_n^{(k)}$ operators that will be useful when apply the renewal theorem.

Lemma 6.2. *For all $k \geq 1$, $n \geq 1$, and $\eta \in \mathbf{L}$,*

$$\|R_n^{(k)} \eta\|_{\mathcal{W}} \leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\eta\|_{\mathcal{W}}, \quad (6.8)$$

$$\|R_n^{(k)} \eta\|_{\mathcal{S}} \leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\eta\|_{\mathcal{S}}, \quad (6.9)$$

$$\|R_n^{(k)} \eta\|_{\mathcal{S}} \leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \left[3 (\beta^a)^k \|\eta\|_{\mathcal{S}} + \|\eta\|_{\mathcal{W}} \right]. \quad (6.10)$$

Proof. We begin by noting the following integral identity which will be used throughout the proof.

Observation 1 For $\eta \in \mathbf{L}$, $t \in [p, q]$, and ψ in \mathbf{B} or \mathbf{A} we have,

$$\int_{\Lambda} S(t) \left[R_n^{(k)}[\eta] \right] \psi d\lambda = \int_{\Lambda} S(t_k) [\eta] \mathbf{1}_{\{r^{(k)}=n\}} \psi \circ T^k d\lambda$$

Verification of this identity is a routine application of eqs. (5.18) and (6.1) once one notes that $\mathbf{1}_{T^k\{r^{(k)}=n\}}$ is constant along unstable curves and thus $S(t)\mathbf{1}_{T^k\{r^{(k)}=n\}} = \mathbf{1}_{T^k\{r^{(k)}=n\}}$.

Claim 1 For all $n \geq 1$, $k \geq 1$, and ψ in \mathbf{A} or \mathbf{B} respectively,

$$\begin{aligned} \left\| \mathbf{1}_{\{r^{(k)}=n\}} \psi \circ T^k \right\|_{\mathbf{A}} &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\psi\|_{\mathbf{A}}, \\ \left\| \mathbf{1}_{\{r^{(k)}=n\}} \psi \circ T^k \right\|_{\mathbf{B}} &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\psi\|_{\mathbf{B}}. \end{aligned}$$

Proof of Claim 1. We will verify the first inequality, the proof of the second is identical. First note that, since return times are independent of the vertical coordinate of a point, $\mathbf{1}_{\{r^k=n\}}$ is constant along vertical lines. We will abuse notation slightly and let $\mathbf{1}_{\{r^k=n\}}$ denote an indicator function on either Λ or on $[p, q]$.

$$\left\| \mathbf{1}_{[r^{(k)}=n]} \psi \circ T^k \right\|_x^1 = \mathbf{1}_{[r^{(k)}=n]}(x) \left\| \psi \circ T^k \right\|_x^1 \leq \mathbf{1}_{[r^{(k)}=n]}(x) \|\psi\|_{u^k(x)}^1$$

An elementary distortion estimate shows that

$$\left\| \frac{u_*^k \mathbf{1}_{[r^{(k)}=n]}}{\mu[r^{(k)}=n]} \right\|_{\sup} \leq \kappa + 1.$$

Integrating yields

$$\begin{aligned} \left\| \mathbf{1}_{[r^{(k)}=n]} \psi \circ T^k \right\|_{\mathbf{A}} &= \int_{[p,q]} \mathbf{1}_{[r^{(k)}=n]}(x) \|\psi\|_{u^k(x)}^1 d\mu(x) \\ &= \int_{[p,q]} u_*^k \mathbf{1}_{[r^{(k)}=n]}(x) \|\psi\|_x^1 d\mu(x) \\ &\leq \mu[r^{(k)}=n] \left\| \frac{u_*^k \mathbf{1}_{[r^{(k)}=n]}(x)}{\mu[r^{(k)}=n]} \right\|_{\sup} \int_{[p,q]} \|\psi\|_x^1 d\mu(x) \\ &\leq \lambda[r^{(k)}=n] [\kappa + 1] \|\psi\|_{\mathbf{A}} \end{aligned}$$

which verifies the claim. ■

Claim 2 For all $n \geq 1, k \geq 1$, and $\eta \in \mathbf{L}$,

$$\begin{aligned} \left\| R_n^{(k)} \eta \right\|_{\mathcal{W}} &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\eta\|_{\mathcal{W}} \\ \left\| R_n^{(k)} \eta \right\|_s &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\eta\|_s \\ \text{Lip}_s \left(R_n^{(k)} \eta \right) &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \beta^k \text{Lip}_s(\eta) \end{aligned}$$

Proof of Claim 2. The proofs of all three of the inequalities above are similar, we will only verify the last. Given $\eta \in \mathbf{L}$, fix $t, s \in [p, q]$ and $\psi \in \mathbf{A}$. We apply Observation 1, Claim 1, and eq. (5.19) in the following computation,

$$\begin{aligned} \frac{\int_{\Lambda} (S(t) - S(s)) R_n^{(k)} \eta \psi d\lambda}{|t - s|} &= \frac{|t_k - s_k|}{|t - s|} \frac{\int_{\Lambda} (S(t_k) - S(s_k)) \eta \mathbf{1}_{[r^{(k)}=n]} \psi \circ T d\lambda}{|t_k - s_k|} \\ &\leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \beta^k \text{Lip}_s(\eta) \|\psi\|_{\mathbf{A}} \end{aligned}$$

Taking a supremum over $t, s \in [p, q]$ and $\psi \in \mathbf{A}$ with $\|\psi\|_{\mathbf{A}} \leq 1$ yields the claimed inequality. ■

Observation 2 Note that for all $n \geq 1, k \geq 1$ and $\eta \in \mathbf{L}$, an application of the third inequality from Claim 2 yields,

$$\begin{aligned} \left\| R_n^{(k)} \eta \right\|_{\mathcal{S}} &= \left\| R_n^{(k)} \eta \right\|_s + \text{Lip}_s \left(R_n^{(k)} \eta \right) \\ &\leq \left\| R_n^{(k)} \eta \right\|_s + \beta^k [\kappa + 1] \lambda \left[r^{(k)} = n \right] \|\eta\|_{\mathcal{S}} \end{aligned}$$

We must bound the first term above.

Claim 3 For all $n \geq 1, k \geq 1$ and $\eta \in \mathbf{L}$,

$$\left\| R_n^{(k)} \eta \right\|_s \leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] \left[2(\beta^a)^k \|\eta\|_s + \|\eta\|_{\mathcal{W}} \right]$$

Proof of Claim 3. Fix $t \in [p, q]$ and $\psi \in \mathbf{B}$ such that $\|\psi\|_{\mathbf{B}} \leq 1$. Define $\psi_0(x, y) = \psi \left(T^k(x, 0) \right)$. Applying eq. (5.12), integrating, and applying Claim 1, we obtain

$$\left\| \mathbf{1}_{[r^{(k)}=n]} \left[\psi \circ T^k - \psi_0 \right] \right\|_{\mathbf{B}} \leq [\kappa + 1] \lambda \left[r^{(k)} = n \right] 2(\beta^a)^k \|\psi\|_{\mathbf{B}}.$$

Also note that ψ_0 is constant along vertical lines and that $\|\psi_0\|_{\mathbf{A}} \leq 1$. Applying

Observation 1 it follows that,

$$\begin{aligned}
\int_{\Lambda} S(t) R_n^{(k)} \eta \psi d\lambda &= \int_{\Lambda} S(t_k) \eta \mathbf{1}_{[r^{(k)}=n]} [\psi \circ T^k - \psi_0] d\lambda \\
&\quad + \int_{\Lambda} S(t_k) \eta \mathbf{1}_{[r^{(k)}=n]} \psi_0 d\lambda \\
&\leq [\kappa + 1] \lambda [r^{(k)} = n] \left[2(\beta^a)^k \|\eta\|_s \|\psi\|_{\mathbf{B}} + \|\eta\|_{\mathcal{W}} \|\psi_0\|_{\mathbf{A}} \right] \\
&\leq [\kappa + 1] \lambda [r^{(k)} = n] \left[2(\beta^a)^k \|\eta\|_s + \|\eta\|_{\mathcal{W}} \right]
\end{aligned}$$

Taking a supremum over t and ψ with $\|\psi\|_{\mathbf{B}} \leq 1$ completes the proof. \blacksquare

By applying the inequalities from Claim 2, Observation 2, and Claim 3 we compute

$$\begin{aligned}
\|R_n^{(k)} \eta\|_{\mathcal{S}} &\leq [\kappa + 1] \lambda [r^{(k)} = n] \left[\beta^k \|\eta\|_s + 2(\beta^a)^k \|\eta\|_s + \|\eta\|_{\mathcal{W}} \right] \\
&\leq [\kappa + 1] \lambda [r^{(k)} = n] \left[3(\beta^a)^k \|\eta\|_s + \|\eta\|_{\mathcal{W}} \right]
\end{aligned}$$

This verifies eq. (6.10). Note that eqs. (6.8) and (6.9) are results of Claim 2. \square

Lemma 6.3. *For all $\eta \in \mathbf{L}$ and $k \geq 1$*

$$\|R(z)^k \eta\|_{\mathcal{W}} \leq [\kappa + 1] |z|^k \|\eta\|_{\mathcal{W}}, \quad (6.11)$$

$$\|R(z)^k \eta\|_{\mathcal{S}} \leq [\kappa + 1] |z|^k \left[3(\beta^a)^k \|\eta\|_{\mathcal{S}} + \|\eta\|_{\mathcal{W}} \right]. \quad (6.12)$$

Proof. We will prove eq. (6.12). The proof of eq. (6.11) is similar.

We note that $\min r^{(k)} \geq 2k$ and apply Lemma 6.2 so that we have

$$\begin{aligned}
\|R(z)^k \eta\|_{\mathcal{S}} &\leq \sum_{n=2k}^{\infty} |z^n| \|R_n^{(k)} \eta\|_{\mathcal{S}} \\
&\leq |z|^k \sum_{n=2k}^{\infty} [\kappa + 1] \lambda \left\{ r^k = n \right\} \left[3(\beta^a)^k \|\eta\|_{\mathcal{S}} + \|\eta\|_{\mathcal{W}} \right] \\
&= [\kappa + 1] |z|^k \left[3(\beta^a)^k \|\eta\|_{\mathcal{S}} + \|\eta\|_{\mathcal{W}} \right].
\end{aligned}$$

\square

Obviously we could have obtained $|z|^{2k}$ as a multiplier in the inequalities above. We opt for the weaker bound as it makes no difference in what follows and is slightly less cumbersome.

Lemma 6.4. *If $|z| \leq 1$ and $\eta \in \mathbf{L}$, then*

$$\|R(z) \eta\|_{\mathbf{L}} \leq \|\eta\|_{\mathbf{L}}. \quad (6.13)$$

Proof. We begin by bounding the sup-norm term in $\|\cdot\|_{\mathbf{L}}$,

$$\begin{aligned}
\|R(z)\eta\|_{\sup} &= \sup_{(x,y) \in \Lambda} \left| \sum_{n=1}^{\infty} z^n R_n \eta(x,y) \right| \\
&= \sup_{(x,y) \in \Lambda} \left| \sum_{n=1}^{\infty} z^n T_* \left(\mathbf{1}_{\{r=n\}} \eta \right) (x,y) \right| \\
&= \sup_{(x,y) \in \Lambda} \left| \sum_{n=1}^{\infty} z^n [\mathbf{1}_{\{r=n\}} \circ T^{-1}](x,y) [\eta \circ T^{-1}](x,y) \right| \\
&= \sup_{(x,y) \in \Lambda} \left| z^{r(T^{-1}(x,y))} [\eta \circ T^{-1}](x,y) \right| \\
&\leq \sup_{(x,y) \in \Lambda} \left| [\eta \circ T^{-1}](x,y) \right| \\
&\leq \|\eta\|_{\sup}
\end{aligned}$$

For the $Lip_u(\cdot)$ -term, fix $(x,y) \in \Lambda$ and $s, t \in [p, q]$, Computing as before we obtain

$$\begin{aligned}
|[S(s) - S(t)] R(z)\eta(x,y)| &= \left| z^{r(T^{-1}(x,y))} [S(s) - S(t)] T_* \eta \right| \\
&\leq |[S(s) - S(t)] T_* \eta| \\
&\leq \beta Lip_u(\eta) |s - t|.
\end{aligned}$$

Since (x,y) , s , and t were arbitrary $Lip_u(R(z)\eta) \leq \beta Lip_u(\eta)$. We conclude that

$$\|R(z)\eta\|_{\mathbf{L}} \leq \|\eta\|_{\sup} + \beta Lip_u(\eta) \leq \|\eta\|_{\mathbf{L}}.$$

□

6.3 Essential Spectrum

Lemma 6.5. *For each $|z| \leq 1$ the operator $R(z): \mathcal{S} \hookrightarrow \mathcal{S}$ is quasi-compact with spectral radius less than or equal to $|z|$ and essential spectral radius less than or equal to $\beta^a |z|$.*

Proof. This follows from Hennion's Theorem [13] in light of the compact embedding proved in Lemma 5.4 and the uniform Lasota-Yorke inequalities proved in Lemma 6.3. □

6.4 Peripheral Spectrum

Lemma 6.6. *For each z with $|z| = 1$,*

1. *The peripheral spectrum of $R(z)$ consists of semi-simple⁴ eigenvalues.*

⁴An eigenvalue is semi-simple if its algebraic and geometric multiplicities match.

2. Every peripheral eigenvector of $R(z)$ is in \mathbf{L} .

Proof. This follows from Lemma 6.6 by a standard argument, and can be found in a slightly different setting in [4] Proposition 3.5. \square

Lemma 6.7. For $z \neq 1$ with $|z| \leq 1$, $I - R(z)$ is invertible. 1 is a simple eigenvalue of $R(1)$ and the associated eigenspace is $\text{span}\{\mathbf{1}_\Lambda\}$.

Proof. The proof of this lemma will be divided into several distinct parts.

Claim 1: For all $|z| \leq 1$ the operator $R(z) - I$ is invertible if and only if 1 is not an eigenvalue of $R(z)$.

Proof of Claim 1. If 1 is an eigenvalue of $R(z)$, then $R(z) - I$ is not invertible by the definition of an eigenvalue. Suppose that $R(z) - I$ is not invertible, then 1 is a point in the spectrum of $R(z)$. By Lemma 6.5 the operator $R(z)$ is quasi-compact with essential spectral radius less than $\beta^a |z|$, which is strictly less than 1, therefore 1 is a point in the spectrum of $R(z)$ that is outside the essential spectrum. It follows that 1 is an eigenvalue of $R(z)$ and that the eigenvector associated to the eigenvalue 1 lies in a finite dimensional $R(z)$ invariant subspace of \mathcal{S} . \blacksquare

Claim 2: If $|z| < 1$, then $R(z) - I$ is invertible.

Proof of Claim 2. Fix z such that $|z| < 1$. It follows from Lemma 6.5 that the spectral radius of $R(z)$ is at most $|z|$. By assumption $|z| < 1$, so 1 is not an eigenvalue of $R(z)$. By the previous claim $R(z) - I$ is invertible. \blacksquare

Claim 3: If $|z| = 1$ and $z \neq 1$, then $I - R(z)$ is invertible. The operator $R(1)$ has a simple eigenvalue at 1 and the associated eigenspace is $\text{span}\{\mathbf{1}_\Lambda\}$.

Proof of Claim 3. We will verify both parts of the claim simultaneously. Let z be a complex number such that $|z| = 1$ and let $\eta \in \mathcal{S}$ be an eigenvector of $R(z)$ with eigenvalue 1, that is

$$R(z)\eta = \eta.$$

The proof relies on two observations about η :

Obs 1 η satisfies the following identity

$$[\eta \circ T](x, y) z^r = \eta(x, y). \quad (6.14)$$

Obs 2 η is a constant multiple of $\mathbf{1}_\Lambda$.

We will verify both observations after completing the proof of Claim 3.

We will show that, if $\eta \neq 0$, then $z = 1$. By Observation 2 η is constant, since T preserves Lebesgue measure $\eta \circ T = \eta$. It follows that eq. (6.14) reduces to

$$(z^{r(x)} - 1)\eta = 0.$$

The equation above is satisfied if $\eta = 0$ or if $z^{r(x)} = 1$.

The equation $z^{r(x)} = 1$ is satisfied if and only if for all $a \in \text{image}(r) \subseteq \mathbb{Z}$,

$$a \frac{\arg(z)}{2\pi} \in \mathbb{Z}.$$

The inclusion above can hold if and only if there exists a rational number b/c such that $\frac{\arg(z)}{2\pi} = b/c$. Assuming that b/c is reduced we see that $ab/c \in \mathbb{Z}$ and if and only if c divides a . Therefore, $\frac{\arg(z)}{2\pi} = b/c$ and c divides a for all $a \in \text{image}(r)$. From section 3 it follows that $\text{image}(r) = \{n \in \mathbb{N} : n \geq 2\}$ and hence the greatest common divisor of $\text{image}(r)$ is 1 so that $c = 1$ and hence $\frac{\arg(z)}{2\pi} \in \mathbb{Z}$. Therefore the principal value of the argument of z is 0 and hence $z = 1$.

T preserves Lebesgue measure on Λ . By eq. (6.6) we have that $R(1)$ is the Frobenius-Perron operator of T . It follows that $R(1)\mathbf{1}_\Lambda = \mathbf{1}_\Lambda$. By Observation 2 any η that satisfies the eigenvector equation $R(1)\eta = \eta$ is a multiple of $\mathbf{1}_\Lambda$. We have verified that $\mathbf{1}_\Lambda$ is a basis for the eigenspace associated to the eigenvalue 1. By Lemma 6.6 the eigenvalue 1 is semi-simple. We conclude that 1 is a simple eigenvalue of $R(1)$.

We have observed that if $R(z)\eta = \eta$, then $\eta \neq 0$ implies that $z = 1$. By contraposition, If $R(z)\eta = \eta$ and $z \neq 1$, then $\eta = 0$. We conclude that for $z \neq 1$, the operator $R(z)$ does not have 1 as an eigenvalue. By our previous claim we conclude that $I - R(z)$ is invertible. ■

To complete the proof of the lemma it remains to verify Observation 1 and Observation 2 from the proof of the last claim.

Observation 1: If $|z| = 1$ and $\eta \in \mathcal{S}$ such that $R(z)\eta = \eta$, then for almost every $(x, y) \in \Lambda$,

$$[\eta \circ T](x, y) z^r = \eta(x, y).$$

Proof of Observation 1. By Lemma 6.6 we have $\eta \in \mathbf{L}$. Since $|\eta|_\infty \leq \|\eta\|_{\text{sup}} \leq$

$\|\eta\|_{\mathbf{L}}$ we have $\eta \in L^\infty(\Lambda, \lambda)$. For all ψ and η in \mathbf{L} we have

$$\begin{aligned} \int R(z)\eta \psi d\lambda &= \int \sum_{n=1}^{\infty} z^n R_n \eta \psi d\lambda = \sum_{n=1}^{\infty} \int z^n T_*(\eta \mathbf{1}_{[r=n]}) \psi d\lambda \\ &= \sum_{n=1}^{\infty} \int \eta z^n \mathbf{1}_{[r=n]} \psi \circ T d\lambda = \int \sum_{n=1}^{\infty} \eta z^n \mathbf{1}_{[r=n]} \psi \circ T d\lambda \\ &= \int \eta z^r \psi \circ T d\lambda. \end{aligned}$$

Since $\eta \in L^\infty(\lambda)$ we have $\eta \in L^2(\lambda)$. Define $\Gamma(z)$ on $L^\infty(\lambda)$ by $W(z)\psi = z^r \psi \circ T$. Now we compute as in [11],

$$\begin{aligned} |W(z)\eta - \eta|_2^2 &= |W(z)\eta|_2^2 - 2\operatorname{Re}\langle W(z)\eta, \eta \rangle + |\eta|_2^2 \\ &= |W(z)\eta|_2^2 - 2\operatorname{Re}\langle \eta, R(z)\eta \rangle + |\eta|_2^2 \\ &= |W(z)\eta|_2^2 - 2\operatorname{Re}\langle \eta, \eta \rangle + |\eta|_2^2 \\ &= |W(z)\eta|_2^2 - |\eta|_2^2, \end{aligned}$$

and note that

$$|W(z)\eta|_2^2 = \int |\eta|^2 \circ T d\lambda = \int |\eta|^2 d\lambda = |\eta|_2^2,$$

from which we conclude that $W(z)\eta = [\eta \circ T] z^r = \eta$ except possibly on a null set.

We have verified eq. (6.14). ■

Observation 2: If $|z| = 1$ and $\eta \in \mathcal{S}$ so that $\mathbb{R}(z)\eta = \eta$, then η is a constant multiple of $\mathbf{1}_\Lambda$.

Proof of Observation 2. We begin by showing that η is essentially constant along stable fibres. For each $j \geq 1$ select $\tau_j \in C^\infty$ such that $|\tau_j - \eta|_1 < 2^{-j}$. Note that $|W(\tau_j - \eta)|_1 = |z^r(\tau_j - \eta) \circ T|_1 = |\tau_j - \eta|_1 < 2^{-j}$. Let $\bar{\tau}_j(x, y) = \int \tau_j(x, y) dy$ and note that by the mean value theorem there exists $s \in (0, 1)$ and $t \in (y, s)$ such that

$$|\tau_j(x, y) - \bar{\tau}_j(x, y)| = |\tau_j(x, y) - \tau_j(x, s)| = |\partial_y \tau_j(x, t)| |y - s| \leq |\partial_y \tau_j|_\infty |y - s|.$$

Further application of the mean value theorem yields

$$|W^n \tau_j(x, y) - W^n \bar{\tau}_j(x, y)| \leq |\partial_y \tau_j|_\infty \left| \partial_y v_x^{(n)} \right|_\infty \leq |\partial_y \tau_j|_\infty \beta^n.$$

For each $j \geq 1$ select $n = n(j)$ such that $|\partial_y \tau_j|_\infty \beta^n + 2^{-j} < 10 \cdot 2^{-j}$ and note that

$$|\eta - \bar{\tau}_j|_1 \leq |W^n \eta - W^n \tau_j|_1 + |W^n \tau_j - W^n \bar{\tau}_j|_1 \leq 10 \cdot 2^{-j}.$$

We see that η is the L^1 -limit of functions that are constant along stable fibres. It follows that for μ -a.e. $x \in [p, q]$,

$$\text{for Leb-a.e. } y, \eta(x, y) = \int_{\ell(x)} \eta(x, z) d\text{Leb}(z), \quad (6.15)$$

Next we will use the unstable regularity of η to show that Property 6.15 holds for every $x \in [p, q]$. To verify this suppose that x failed to satisfy Property 6.15. This can happen if and only if there exist sets $A_x, B_x \subset \ell(x)$ and $\epsilon > 0$, such that $\text{Leb}(A_x) > 0$, $\text{Leb}(B_x) > 0$, and for all y in A_x and z in B_x

$$\eta(x, y) - \eta(x, z) \geq \epsilon. \quad (6.16)$$

For $w \neq x$ let $A_w \subset \ell(w)$ be the set obtained by sliding⁵ A_x along unstable disks into $\ell(w)$ and let B_w be defined similarly. Note that $\text{Leb}(A_w) > 0$ if and only if $\text{Leb}(A_x) > 0$. Since η is in \mathbf{L} we have that

$$|\eta(x, y) - \eta(\ell(w) \cap \gamma(x, y))| \leq \text{Lip}_u(\eta) |x - w|.$$

Choose $\delta > 0$ so that $\text{Lip}_u(\eta) \delta < \epsilon/3$. Fix $w \in [p, q]$ such that $|w - x| < \delta$. Select $(w, y) \in A_w$ and $(w, z) \in B_w$ and let $(x, y') \in A_x$ and $(x, z') \in B_x$ denote the points obtained by sliding along unstable disks back to $\ell(x)$. We compute,

$$\eta(w, y) - \eta(w, z) \geq \eta(x, y') - \eta(x, z') - 2\text{Lip}_u(\eta) |x - w| \geq \epsilon - 2\text{Lip}_u(\eta) \delta \geq \frac{\epsilon}{3}.$$

We have just shown that for every $w \in [p, q]$ with $|w - x| < \delta$ Property 6.16 holds at w , thus Property 6.15 fails at w . This contradicts our observation that eq. (6.15) holds for μ -a.e. $x \in [p, q]$. We conclude that eq. (6.15) holds for every $x \in [p, q]$.

Define $h(x) = \int_0^1 \eta(x, y) dy$. This function is Lipschitz. To verify this fix $x, w \in [p, q]$. Let $A_x \subset \ell(x)$ denote the set of points in $\ell(x)$ where eq. (6.15) fails and let A_w be defined similarly. By the previous paragraph both A_x and A_w are null sets. Let $B \subset \ell(x)$ be the set obtained by sliding A_w along unstable disks into $\ell(x)$. The set B is a null set, therefore the set $G = \ell(x) - (A_x \cup B)$ consisting of points in $\ell(x)$ where $\eta(x, y) = h(x)$ and $\eta(\gamma(x, y) \cap \ell(w)) = h(w)$ has full measure. Choose $(x, y) \in G$ and note that

$$|h(x) - h(w)| = |\eta(x, y) - \eta(\gamma(x, y) \cap \ell(w))| \leq \text{Lip}_u(\eta) |x - w|,$$

so h is Lipschitz with Lipschitz constant at most $\text{Lip}_u(\eta)$.

Next we would like to verify $\int [W(z)\eta](x, y) dy = z^r [h \circ u](x)$. Note that T maps $\ell(x)$ into $\ell(u(x))$ affinely. We will apply the change of variable $y' =$

⁵By sliding along unstable disks we mean $(x, y) \mapsto \gamma(x, y) \cap \ell(w)$

$g_x(y)$ noting that $dy' = \partial_y g_x(y) dy$ and that $\partial_y g_x(y)$ is constant and exactly equal to the length of the interval $T\ell(x) \subset \ell(u(x))$

$$\int_0^1 z^{r(x)} (\eta \circ T)(x, y) dy = z^{r(x)} \frac{1}{|T\ell(x)|} \int_{T\ell(x)} \eta(u(x), y') dy' = z^{r(x)} h(u(x))$$

Applying Observation 1 we obtain

$$z^r [h \circ u](x) = h(x) \quad (6.17)$$

Next we deduce that h is an essentially constant function. We will apply Corollary 3.2 from [2]. We reformulate the Corollary in our notation for the convenience of the reader.

Suppose that:

- $u: [p, q] \rightarrow \mathbb{C}$ is a probability preserving, almost onto Gibbs-Markov map with respect to the partition $\alpha = \{I_j, I'_j : j = 2, \dots, \infty\}$ ⁶.
- $\varphi: [p, q] \rightarrow \{z \in \mathbb{C} : |z| = 1\}$ is α -measurable.
- $h: [p, q] \rightarrow \{z \in \mathbb{C} : |z| = 1\}$ is Boreal measurable and $\varphi(x) = h \cdot \bar{h} \circ u$

Then h is essentially constant.

Let us verify that u satisfies the first hypothesis of the Corollary. For each $a \in \alpha$ the map $u|_a$ is a homeomorphism onto $[p, q]$ with C^2 inverse $v_a: [p, q] \rightarrow a$. The map u is uniformly expanding by Lemma 4.1 and satisfies Adler's bounded distortion property by Lemma 4.2. By Example 2 of [2] it follows that u is a mixing Gibbs-Markov map. Since every branch of u is onto, u is almost onto as defined immediately after Theorem 3.1 of [2].

Since u is a Gibbs-Markov map, u is ergodic. Taking the complex modulus of eq. (6.17) yields $|h| = |h \circ u| = |h| \circ u$, thus $|h|$ is an essentially constant function. Since h is Lipschitz, we have that $|h|$ is Lipschitz and therefore pointwise constant. Without loss of generality assume that $|h| = 1$.

Since h is a circle valued function we have $\bar{h} = 1/h$. Let $\varphi(x) = h \cdot \bar{h} \circ T$. By eq. (6.17) we have

$$\varphi(x) = h \cdot \bar{h} \circ T = \frac{h}{h \circ T} = z^{r(x)}.$$

Since $r(x)$ is measurable with respect to the partition α we have that φ is circle valued and α -measurable. We have just verified that φ satisfies the second hypothesis above and that h and φ are related as required in the third hypothesis

⁶see section 3

by definition.

Applying the Corollary we see that h is essentially constant. Since h is Lipschitz we conclude that h is pointwise constant. Let h_0 denote the constant value of h .

Define $H(x, y) = h_0$, this function is clearly in \mathbf{L} . On each vertical line the function H agrees with η except possibly on a set of one dimensional Lebesgue measure zero. It follows that for all $t \in [p, q]$ there exists a λ -null set N_t such that for all $(x, y) \in \Lambda - N_t$ we have $S(t)\eta - H(x, y) = 0$. With this fact it follows directly from eqs. (5.23) and (5.24) that $\|\eta - H\|_{\mathcal{S}} = 0$ and $\text{Lip}_{\mathcal{S}}(\eta - H) = 0$, thus $\|\eta - H\|_{\mathcal{S}} = 0$. We conclude that η and H are in the same \mathcal{S} -equivalence class. \blacksquare

Having verified Observation 1 and Observation 2 from the proof of Claim 3 we see that the lemma follows by combining Claim 2 and Claim 3. \square

7 Rate of Decay of Correlation

Proof of Theorem 1.1. We will apply [11] Theorem 1.1.

The renewal equation hypothesis of the theorem is checked in Lemma 6.1. Note that $\sum_n \lambda[r = n] = \lambda(\Lambda) = 1$. Set $k = 1$ in eq. (6.9) and sum both sides of the inequality to see that $\sum \|R_n\| < \infty$. The spectral gap and aperiodicity hypothesis of the theorem are verified in Lemma 6.7. By eq. (4.10) and eq. (6.9) with $k = 1$, we see that $\|R_n\| \approx \left(\frac{1}{n}\right)^{1/\alpha+2}$, therefore

$$\sum_{k>n} \|R_k\| = O\left(\left(\frac{1}{n}\right)^{1/\alpha+1}\right).$$

From Lemma 6.7 we see that the spectral projector for the eigenvalue 1 associated to $R(1)$ is

$$P\eta = \mathbf{1}_{\Lambda} \int_{\Lambda} \eta d\lambda.$$

It follows that for any $\eta \in \mathbf{L}$

$$P \frac{dR}{dz}(1) P\eta = \sum_{n=1}^{\infty} n \lambda[r = n] P\eta.$$

So by Kac's Lemma $P \frac{dR}{dz}(1) P = \frac{1}{\text{Leb}(\Lambda)} P$. Similarly we see that

$$P_k \eta = \sum_{l>k} P R_l P\eta = P\eta \sum_{l>k} \lambda[r = l] = \lambda[r > k] P\eta.$$

From Theorem 1.1 of [11] we obtain the expansion

$$B_n = \text{Leb}(\Lambda)P + \text{Leb}(\Lambda)^2 \sum_{k>n} P_k + E_n$$

where

$$\|E_n\| = \begin{cases} O\left(\left(\frac{1}{n}\right)^{1+1/\alpha}\right), & \text{if } \alpha > 1; \\ O\left(\frac{\log(n)}{n^2}\right), & \text{if } \alpha = 1; \\ O\left(\left(\frac{1}{n}\right)^{2/\alpha}\right), & \text{if } \alpha < 1. \end{cases}$$

Recalling eq. (4.4) we see that

$$B_n \eta = \mathbf{1}_\Lambda \int_\Lambda \eta d\text{Leb} + \mathbf{1}_\Lambda \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta d\text{Leb} + E_n \eta.$$

If η and ψ are Lipschitz on the square, then $\mathbf{1}_\Lambda \eta \in \mathbf{L}$ and we obtain

$$\int B_n \eta \psi d\text{Leb} = \int \mathbf{1}_\Lambda B_*^n (\mathbf{1}_\Lambda \eta) \psi d\text{Leb} = \int \mathbf{1}_\Lambda \eta (\mathbf{1}_\Lambda \psi) \circ B^n d\text{Leb}$$

If η and ψ are the restrictions to Λ of Lipschitz functions on the square, then

$$\begin{aligned} \int_\Lambda \eta \psi \circ B^n d\text{Leb} &= \int_\Lambda \eta d\text{Leb} \int_\Lambda \psi d\text{Leb} + \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta d\text{Leb} \int_\Lambda \psi d\text{Leb} \\ &\quad + \int_\Lambda E_n \eta \psi d\text{Leb}. \end{aligned}$$

Note that $\sum_{k>n} \text{Leb}[r > k] \approx \left(\frac{1}{n}\right)^{1/\alpha}$ and that regardless of the value of α this decays slower than $\|E_n\|$. If $\int \eta \neq 0$ and $\int \psi \neq 0$, then

$$\begin{aligned} \int_\Lambda \eta \psi \circ B^n d\text{Leb} - \int_\Lambda \eta d\text{Leb} \int_\Lambda \psi d\text{Leb} &= \sum_{k>n} \text{Leb}[r > k] \int_\Lambda \eta d\text{Leb} \int_\Lambda \psi d\text{Leb} \\ &\quad + \int_\Lambda E_n \eta \psi d\text{Leb} \\ &\approx \left(\frac{1}{n}\right)^{1/\alpha}. \end{aligned} \tag{7.1}$$

For functions with integral zero the rate of decay may be faster than $\left(\frac{1}{n}\right)^{1/\alpha}$. \square

Corollary 7.0.1. *If the hypotheses of Theorem 1.1 are satisfied and additionally either $\int_\Lambda \psi = 0$ or $\int_\Lambda \eta = 0$, then $\text{Cor}(k, \psi, \eta, B)$ is a summable sequence.*

8 The Perturbed Renewal Operator

In this section we will select an observable $X: [0, 1]^2 \rightarrow \mathbb{R}$ such that $\int X = 0$ and deduce the distributional limit behavior of sequences of the form

$$\frac{1}{A_n} \sum_{k=0}^{n-1} X \circ B^k,$$

where A_n is a carefully chosen increasing sequence of normalizing constants and B is an IBT with contact coefficients c_j and contact exponents α_j , where $j = 0$ or $j = 1$.

In order to study the distributional behavior of X we will follow [10] and study a perturbed version of the renewal operator defined in eq. (6.4). Before we can apply renewal methods we must recast the problem in terms of the induced map T defined in section 4. We define an observable $\xi: \Lambda \rightarrow \mathbb{R}$ derived from the observable X as follows,

$$\xi(x, y) = \sum_{k=0}^{r(x)-1} (X \circ B^k)(x, y). \quad (8.1)$$

We define perturbed versions of the first return time operators R_n defined in eq. (6.1) and the renewal operator $R(z)$ defined in eq. (6.4). For all $t \in \mathbb{R}$, let

$$R_n(t)\eta = R_n[\exp(it\xi)\eta] \quad (8.2)$$

and

$$R(z, t) = \sum_{n=1}^{\infty} z^n R_n(t) \quad (8.3)$$

8.1 Asymptotics for ξ

First we will investigate the asymptotic behavior of ξ on the set of points returning in $n + 2$ steps.

Lemma 8.1. *Suppose that $X: [0, 1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0, 1]$ and that $(x, y) \in \Lambda$ is a point such that $x \in [A, q]$ and $r(x, y) = n + 2$ for some $n \geq 0$. As $n \rightarrow \infty$,*

$$\xi(x, y) = n \int_0^1 X(0, y^{1+\frac{1}{\alpha_0}}) dy + O(n^{1-\gamma}) + O(n^{1-\frac{\gamma}{\alpha_0}}).$$

If $x \in [p, A]$, then as $n \rightarrow \infty$,

$$\xi(x, y) = n \int_0^1 X(1, y^{1+\frac{1}{\alpha_1}}) dy + O(n^{1-\gamma}) + O(n^{1-\frac{\gamma}{\alpha_1}}).$$

Proof. We will prove the first asymptotic expansion, the proof of the second is similar. Through out this proof we will suppress the subscript on the contact parameters ($\alpha = \alpha_0$ and $c = c_0$). By eq. (8.1)

$$\xi(x, y) = \sum_{k=0}^{n+1} X(x_k, y_k).$$

Since X is γ -Hölder,

$$|X(x_k, y_k) - X(0, y_k)| = O(x_k^\gamma) = O\left(n^{-\frac{\gamma}{\alpha}}\right).$$

$$\left| X(0, y_k) - X\left(0, \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\alpha}}\right) \right| = O(n^{-\gamma}).$$

An end point approximation to the Riemann sum shows that

$$\left| \int_0^1 X\left(0, y^{1+\frac{1}{\alpha}}\right) dy - \frac{1}{n} \sum_{k=1}^{n-1} X\left(0, \left(1 - \frac{k+1}{n}\right)^{1+\frac{1}{\alpha}}\right) \right| = O(n^{-\gamma})$$

A standard triangle inequality argument shows that

$$\xi(x, y) = n \int_0^1 X(0, y^{1+\frac{1}{\alpha}}) dy + O(n^{1-\gamma}) + O(n^{1-\frac{\gamma}{\alpha}})$$

and therefore the claimed asymptotic holds. \square

For convenience we define

$$M_0 = \int_0^1 X(0, y^{1+\frac{1}{\alpha_0}}) dy,$$

$$M_1 = \int_0^1 X(1, y^{1+\frac{1}{\alpha_1}}) dy.$$

Next we investigate the cumulative distribution function of ξ .

Lemma 8.2. *Suppose that $X: [0, 1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0, 1]$.*

- *If $M_0 > 0$, then for t sufficiently large,*

$$\begin{aligned} \lambda([\xi > t] \cap [A, q]) &\sim \frac{M_0}{\alpha_0 \text{Leb}(\Lambda)} \left(\frac{M_0(\alpha_0+1)}{c_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left(\frac{1}{t} \right)^{1+\frac{1}{\alpha_0}}, \\ \lambda([\xi < -t] \cap [A, q]) &= 0. \end{aligned}$$

- *If $M_0 < 0$, then for t sufficiently large,*

$$\begin{aligned} \lambda([\xi > t] \cap [A, q]) &= 0, \\ \lambda([\xi < -t] \cap [A, q]) &\sim \frac{|M_0|}{\alpha_0 \text{Leb}(\Lambda)} \left(\frac{|M_0|(\alpha_0+1)}{c_0 \alpha_0} \right)^{\frac{1}{\alpha_0}} \left(\frac{1}{t} \right)^{1+\frac{1}{\alpha_0}}. \end{aligned}$$

- If $M_1 > 0$, then for t sufficiently large,

$$\begin{aligned}\lambda([\xi > t] \cap [p, A]) &\sim \frac{M_1}{\alpha_1 \text{Leb}(\Lambda)} \left(\frac{M_1(\alpha_1+1)}{c_1 \alpha_1} \right)^{\frac{1}{\alpha_1}} \left(\frac{1}{t} \right)^{1+\frac{1}{\alpha_1}}, \\ \lambda([\xi < -t] \cap [p, A]) &= 0.\end{aligned}$$

- If $M_1 < 0$, then for t sufficiently large,

$$\begin{aligned}\lambda([\xi > t] \cap [p, A]) &= 0, \\ \lambda([\xi < -t] \cap [p, A]) &\sim \frac{|M_1|}{\alpha_1 \text{Leb}(\Lambda)} \left(\frac{|M_1|(\alpha_1+1)}{c_1 \alpha_1} \right)^{\frac{1}{\alpha_1}} \left(\frac{1}{t} \right)^{1+\frac{1}{\alpha_1}}.\end{aligned}$$

Proof. We will prove the first asymptotic, the proofs of the others are similar. We will suppress subscripts through out this proof ($M = M_0$, $\alpha = \alpha_0$, and $c = c_0$). For convenience define for any function f on Λ and real number t , $U(f, t) = [f > t] \cap [A, q]$. Note that by eq. (3.7), $\lambda(U(r, n)) = \frac{p_n^\circ - A}{q - p} = \frac{p_n^\circ - A}{\text{Leb}(\Lambda)}$, thus by eq. (3.15)

$$\lambda(U(r, t)) \sim \frac{1}{\alpha \text{Leb}(\Lambda)} \left(\frac{(\alpha+1)}{c\alpha} \right)^{\frac{1}{\alpha}} \left(\frac{t}{[t]} \right)^{1+\frac{1}{\alpha}} \left(\frac{1}{t} \right)^{1+\frac{1}{\alpha}}.$$

Let $g(x, y) = \xi(x, y) - Mr(x, y)$, then Fix $\epsilon > 0$ and note that,

$$\begin{aligned}\lambda(U(\xi, t)) &\geq \lambda(U(Mr, t(1+\epsilon))) - \lambda(U(|g|, \epsilon t)), \\ \lambda(U(\xi, t)) &\leq \lambda(U(Mr, t(1-\epsilon))) + \lambda(U(|g|, \epsilon t)).\end{aligned}$$

Note that $|g| > \epsilon t$ iff $r > \frac{r}{|g|} \epsilon t$. By Lemma 8.1 $|g| = o(r(x, y))$, thus the quantity $\frac{r}{|g|} \epsilon$ is unbounded as $r \rightarrow \infty$. We conclude that as $t \rightarrow \infty$,

$$\lambda(U(|g|, \epsilon t)) = o(\lambda(U(r, t))).$$

Therefore as $t \rightarrow \infty$,

$$|\lambda(U(\xi, t)) - \lambda(U(r, \frac{t}{M}))| = o(\lambda[r > t]).$$

The claimed asymptotic for $\lambda(U(\xi, t))$ follows, since

$$\left(\frac{t}{M} \left\lfloor \frac{M}{t} \right\rfloor \right)^{1+\frac{1}{\alpha}} = 1 + o(1)$$

as $t \rightarrow \infty$.

It is not hard to check that ξ is continuous on each set $[r = n + 2] \cap [A, q]$ for $n \geq 0$. By Lemma 8.1, for $(x, y) \in [r = n + 2] \cap [A, q]$,

$$\xi(x, y) = Mn + O(n^{1-\gamma}) + O(n^{1-\frac{\gamma}{\alpha}}).$$

For n sufficiently large the first term dominates the last two and ξ is strictly positive on $[r = n + 2] \cap [A, q]$. This leaves finitely many sets where ξ may be negative, on each ξ is continuous, therefore ξ is bounded below. We conclude that, for t sufficiently large,

$$\lambda([\xi < -t] \cap [A, q]) = 0.$$

□

For convenience we define

$$C_0 = \frac{|M_0|}{\alpha_0 \text{Leb}(\Lambda)} \left(\frac{|M_0|(\alpha_0 + 1)}{c_0 \alpha_0} \right)^{\frac{1}{\alpha_0}},$$

$$C_1 = \frac{|M_1|}{\alpha_1 \text{Leb}(\Lambda)} \left(\frac{|M_1|(\alpha_1 + 1)}{c_1 \alpha_1} \right)^{\frac{1}{\alpha_1}}.$$

8.2 Continuity of $R_n(t)$

Next we investigate the continuity of $R_n(t)$ at $t = 0$.

Lemma 8.3. *The operator valued function $R_n(t)$ is continuous at $t = 0$. Further as $t \rightarrow 0$,*

$$\|R(z, t) - R(z, 0)\|_{\mathcal{S}} = O(|t|).$$

Proof. We will show that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $t \in (-\delta, \delta)$ and for all $\eta \in \mathbf{L}$,

$$\|[R_n(t) - R_n(0)]\eta\|_{\mathcal{S}} \leq \epsilon \|\eta\|_{\mathcal{S}}.$$

First note that,

$$[R_n(t) - R_n(0)]\eta = T_* \left((\exp(it\xi) - 1) \mathbf{1}_{[r=n]}\eta \right)$$

Second note that,

$$\|[R_n(t) - R_n(0)]\eta\|_{\mathcal{S}} = \|[R_n(t) - R_n(0)]\eta\|_s + \text{Lip}_s([R_n(t) - R_n(0)]\eta).$$

Fix $\epsilon > 0$. We will estimate the two terms on the right which we label I and II respectively. Let $\psi \in \mathbf{B}$ be a test function with $\|\psi\|_{\mathbf{B}} \leq 1$, v be a point in $[p, q]$, and consider a typical integral from the definition of $\|[R_n(t) - R_n(0)]\eta\|_s$,

$$I = \int_{\Lambda} S(v) T_* \left((\exp(it\xi) - 1) \mathbf{1}_{[r=n]}\eta \right) d\lambda$$

We claim that the facts below follow easily from the definitions.

A. For each $v \in [p, q]$ the operator $S(v)$ is multiplicative, that is for any functions f and g

$$S(v)[fg] = S(v)[f]S(v)[g].$$

Similarly, T_* is multiplicative.

B. For all $n \geq 1$ and $v \in [p, q]$,

$$S(v)T_*\mathbf{1}_{[r=n]} = T_*\mathbf{1}_{[r=n]}.$$

C. Recall the commutation relation eq. (5.18). For all $v \in [p, q]$

$$S(v)T_* = T_*S(v_1),$$

where the value of $v_1: \Lambda \rightarrow [p, q]$ is the first coordinate of $T^{-1}(\gamma(x, y) \cap \ell(v))$.
Note that $v_1(x, y)$ is constant on $T[r = n]$ and takes its value in $[r = n]$.

With the facts above in mind we compute as follows,

$$\begin{aligned} I &= \int_{\Lambda} S(v)T_* \left[(\exp(it\xi) - 1) \mathbf{1}_{[r=n]} \eta \right] \psi d\lambda \\ &= \int_{\Lambda} S(v)T_* \left[(\exp(it\xi) - 1) \eta \right] S(v)T_* \left[\mathbf{1}_{[r=n]} \right] \psi d\lambda && \text{(By A)} \\ &= \int_{\Lambda} S(v)T_* \left[(\exp(it\xi) - 1) \eta \right] T_* \left[\mathbf{1}_{[r=n]} \right] \psi d\lambda && \text{(By B)} \\ &= \int_{\Lambda} S(v)T_* \left[(\exp(it\xi) - 1) \eta \right] \mathbf{1}_{T[r=n]} \psi d\lambda \\ &= \int_{T[r=n]} S(v)T_* \left[(\exp(it\xi) - 1) \eta \right] \psi d\lambda \\ &= \int_{T[r=n]} T_*S(v_1) \left[(\exp(it\xi) - 1) \eta \right] \psi d\lambda && \text{(By C)} \\ &= \int_{\Lambda} S(v_1) \left[(\exp(it\xi) - 1) \eta \right] (\psi \circ T) \mathbf{1}_{[r=n]} d\lambda \\ &= \int_{\Lambda} S(v_1) [\eta] S(v_1) [(\exp(it\xi) - 1)] (\psi \circ T) \mathbf{1}_{[r=n]} d\lambda && \text{(By B)} \\ &\leq \|\eta\|_s \left\| S(v_1) [(\exp(it\xi) - 1)] (\psi \circ T) \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}}. \end{aligned}$$

Note that by fact C above the point v_1 is in the set $[r = n]$.

Similarly, we consider a typical integral from the definition of $Lip_s((R_n(t) - R_n(0))[\eta])$.
Let w be a second point in $[p, q]$ such that $w \neq v$,

$$\begin{aligned} II &= \frac{1}{|v - w|} \int_{\Lambda} (S(v) - S(w)) T_* \left[(\exp(it\xi) - 1) \mathbf{1}_{[r=n]} \eta \right] \psi d\lambda \\ &= \frac{1}{|v - w|} \int_{\Lambda} (S(v_1) - S(w_1)) [\eta] (S(v_1) - S(w_1)) [(\exp(it\xi) - 1)] (\psi \circ T) \mathbf{1}_{[r=n]} d\lambda \\ &\leq Lip_s(\eta) \left\| (S(v_1) - S(w_1)) [\exp(it\xi) - 1] (\psi \circ T) \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}} \\ &\leq \|\eta\|_s \left\| S(v_1) [\exp(it\xi) - 1] (\psi \circ T) \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}} \\ &\quad + \|\eta\|_s \left\| S(w_1) [\exp(it\xi) - 1] (\psi \circ T) \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}} \end{aligned}$$

where as before both v_1 and w_1 are in the set $[r = n]$.

Next we will bound $\left\| S(v_1) [\exp(it\tilde{\xi}) - 1] (\psi \circ T) \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}}$. As before we collect a few easily verified facts,

a. Recall eqs. (5.6) and (5.7) and note that for any functions f , and g ,

$$\|fg\|_x^1 \leq \|f\|_x^1 \|g\|_x^1 + \|f\|_x^1 \|g\|_x^1 = 2\|f\|_x^1 \|g\|_x^1.$$

b. For all $w \in [p, q]$ and $t \in \mathbb{R}$, $S(w) (\exp(it\tilde{\xi}) - 1) = \exp(itS(w)\tilde{\xi}) - 1$. For $r, a \in \mathbb{R}$ we have the following bound on the complex modulus $|\exp(ita) - 1| \leq |t| |a|$. Combining these observations and the definitions from section 5.2 we have for all $x, w \in [p, q]$,

$$\begin{aligned} \|S(w) (\exp(it\tilde{\xi}) - 1)\|_x^1 &\leq |t| \|S(w)\tilde{\xi}\|_x^1 \\ &= |t| \|\tilde{\xi}\|_w^1. \end{aligned}$$

Note the switch from $\|\cdot\|_x^1$ to $\|\cdot\|_w^1$.

Keeping the facts above in mind we compute as follows,

$$\begin{aligned} \|S(w) (\exp(it\tilde{\xi}) - 1) [\psi \circ T] \mathbf{1}_{[r=n]}\|_x^1 &\leq 2\|S(w) (\exp(it\tilde{\xi}) - 1)\|_x^1 \|\psi \circ T\|_{[r=n]}^1 \\ &\quad \text{(By a)} \\ &\leq 2|t| \|\tilde{\xi}\|_w^1 \|\psi \circ T\|_{[r=n]}^1 \quad \text{(By b)} \end{aligned}$$

Recall eq. (5.8) and Claim 1 from the proof of Lemma 6.2. We compute as follows,

$$\begin{aligned} \left\| S(v_1) (\exp(it\tilde{\xi}) - 1) [\psi \circ T] \mathbf{1}_{[r=n]} \right\|_{\mathbf{B}} &= \int_{[p,q]} \|S(v_1) (\exp(it\tilde{\xi}) - 1) [\psi \circ T] \mathbf{1}_{[r=n]}\|_x^1 d\mu(x), \\ &\leq \lambda [r = n] 2(\kappa + 1) |t| \|\tilde{\xi}\|_{v_1}^1 \|\psi\|_{\mathbf{B}} \\ &\leq \lambda [r = n] 2(\kappa + 1) |t| \|\tilde{\xi}\|_{v_1}^1. \end{aligned}$$

As noted above $v_1 \in [r = n]$. By Lemma 8.1, $\|\tilde{\xi}\|_x^1$ is uniformly bounded on $[r = n]$ by some constant $M(n)$. Chose $\delta > 0$ such that $\delta(\kappa + 1)M(n) \|\eta\|_s < \epsilon/3$. It follows from the calculations above that for $|t| < \delta$ we have $I < \epsilon/3$ and $II < 2\epsilon/3$, which completes the proof of continuity.

By the estimates above,

$$\|R_n(t) - R_n(0)\|_{\mathcal{S}} = O\left(|t| \lambda [r = n] \|\tilde{\xi}\|_{v_1}^1\right).$$

For n sufficiently large $\|\xi\|_{v_1}^1 = O(n)$ and $\lambda[r = n] = O\left(n^{-2-\frac{1}{\alpha}}\right)$, thus

$$\|R_n(t) - R_n(0)\|_{\mathcal{S}} = O\left(|t| n^{-1-\frac{1}{\alpha}}\right).$$

The estimate above is summable in n . The result for $R(z, t)$ follows by an easy application of the triangle inequality and monotone convergence. \square

8.3 Expansion of the Dominant Eigenvalue

We begin by providing some sufficient conditions for ξ to be in L^2 .

Lemma 8.4. *Suppose that $X: [0, 1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0, 1]$. If for $j = 0$ and $j = 1$, one of the following conditions is satisfied,*

- i. $\alpha_j < 1$,
- ii. $M_j = 0$ and $\alpha_j \leq 1$,
- iii. $M_j = 0$, $1 < \alpha < 3$, and $\gamma > \frac{\alpha-1}{2}$,

then $\xi \in L^2$.

Proof. Let $I(0, n) = [r = n + 2] \cap [A, q]$ and $I(1, n) = [r = n + 2] \cap [p, A]$, and note that $\{I(j, n) : j \in \{0, 1\}, n \in \mathbb{N}\}$ is a partition of Λ . By the monotone convergence theorem and Hölder's inequality

$$\begin{aligned} \int_{\Lambda} |\xi|^2 d\lambda &= \lim_{k \rightarrow \infty} \sum_{n=0}^k \sum_{j=0}^1 \int_{\Lambda} \mathbf{1}_{I(j, n)} |\xi|^2 d\lambda \\ &\leq \lim_{k \rightarrow \infty} \sum_{n=0}^k \sum_{j=0}^1 \left\| \mathbf{1}_{I(j, n)} |\xi|^2 \right\|_{\sup} \lambda(I(j, n)). \end{aligned}$$

By eq. (3.7) we have $\lambda(I(0, n)) = p_{n+1}^\circ - p_{n+2}^\circ$ and $\lambda(I(1, n)) = q_{n+2}^\circ - q_{n+1}^\circ$. By Lemma 3.3, as $n \rightarrow \infty$

$$\lambda(I(j, n)) = O\left(\left(\frac{1}{n}\right)^{2+\frac{1}{\alpha_j}}\right).$$

- i. By Lemma 8.1, as $n \rightarrow \infty$

$$\left\| \mathbf{1}_{I(j, n)} |\xi|^2 \right\|_{\sup} = O(n^2).$$

Therefore,

$$\left\| \mathbf{1}_{I(j, n)} |\xi|^2 \right\|_{\sup} \lambda(I(j, n)) = O\left(\left(\frac{1}{n}\right)^{\frac{1}{\alpha_j}}\right).$$

If $\alpha_j < 1$, then these terms are summable.

ii. Suppose that $M_j = 0$ and $\alpha \leq 1$. By Lemma 8.1, as $n \rightarrow \infty$

$$\left\| \mathbf{1}_{I(j,n)} |\xi|^2 \right\|_{\sup} = O(n^{2-2\gamma}).$$

Therefore,

$$\left\| \mathbf{1}_{I(j,n)} |\xi|^2 \right\|_{\sup} \lambda(I(j,n)) = O\left(\left(\frac{1}{n}\right)^{2\gamma + \frac{1}{\alpha_j}}\right).$$

Since $2\gamma + \frac{1}{\alpha_j} > 1$, these terms are summable.

iii. Suppose that $M_j = 0$, $1 < \alpha_j < 3$, and $\gamma > \frac{\alpha_j - 1}{2}$. By Lemma 8.1, as $n \rightarrow \infty$

$$\left\| \mathbf{1}_{I(j,n)} |\xi|^2 \right\|_{\sup} = O\left(n^{2-2\frac{\gamma}{\alpha_j}}\right).$$

Therefore,

$$\left\| \mathbf{1}_{I(j,n)} |\xi|^2 \right\|_{\sup} \lambda(I(j,n)) = O\left(\left(\frac{1}{n}\right)^{\frac{2\gamma+1}{\alpha_j}}\right)$$

Since $\frac{2\gamma+1}{\alpha} > 1$, these terms are summable.

□

The following eigenvalue expansions are the key to applying [10] to obtain limit laws.

Lemma 8.5. *Let $\chi(t)$ denote the eigenvalue near 1 of the operator $R(1, t)$ for small t . Suppose that $X: [0, 1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0, 1]$ and $\int_{[0,1]^2} X d\text{Leb} = 0$.*

i. If $\xi \in L^2$, then as $t \rightarrow 0$,

$$\chi(t) \sim 1 - \frac{1}{2}\sigma^2 t^2,$$

where

$$\sigma^2 = \int_{\Lambda} |\xi|^2 d\lambda + 2 \sum_{k=1}^{\infty} \int_{\Lambda} \xi \circ T^k \xi d\lambda.$$

ii. If $\alpha_0 > \alpha_1$, $\alpha_0 > 1$, and $M_0 > 0$, then as $t \rightarrow 0$

$$\begin{aligned} \chi(t) &\sim 1 - A |t|^{1+\frac{1}{\alpha_0}} + iB \operatorname{sgn}(t) |t|^{1+\frac{1}{\alpha_0}}, \\ A &:= C_0 \Gamma\left(-\frac{1}{\alpha_0}\right) \cos\left(\frac{1+\frac{1}{\alpha_0}}{2} \pi\right), \\ B &:= C_0 \Gamma\left(-\frac{1}{\alpha_0}\right) \sin\left(\frac{1+\frac{1}{\alpha_0}}{2} \pi\right). \end{aligned}$$

⁷In particular if the hypotheses of Lemma 8.4 are satisfied

iii. If $\alpha_0 = \alpha_1 =: \alpha$, $\alpha > 1$, $M_0 > 0$ and $M_1 < 0$, then as $t \rightarrow 0$,

$$\begin{aligned}\chi(t) &\sim 1 - A |t|^{1+\frac{1}{\alpha}} + iB \operatorname{sgn}(t) |t|^{1+\frac{1}{\alpha}}, \\ A &:= (C_0 + C_1) \Gamma\left(-\frac{1}{\alpha}\right) \cos\left(\frac{1+\frac{1}{\alpha}}{2}\pi\right), \\ B &:= (C_0 - C_1) \Gamma\left(-\frac{1}{\alpha}\right) \sin\left(\frac{1+\frac{1}{\alpha}}{2}\pi\right).\end{aligned}$$

iv. Suppose that $\alpha_0 = \alpha_1 = 1$, $M_0 \neq 0$, and $M_1 \neq 0$, then as $t \rightarrow 0$,

$$\chi(t) \sim 1 + (C_0 + C_1) |t|^2 \log |t|.$$

Proof. By Lemma 8.3 we have

$$\|R(z, t) - R(z, 0)\|_{\mathcal{S}} = O(|t|).$$

If $e(t)$ is the eigenfunction of $R(1, t)$ associated to the eigenvalue $\chi(t)$ with integral 1, then because eigenvectors depend holomorphically on operators

$$\|e(t) - 1\|_{\mathcal{S}} = O(\|R(z, t) - R(z, 0)\|_{\mathcal{S}}) = O(|t|).$$

- i. By arguments similar to [10] Theorem 3.7 we obtain the claimed expansion. Since T_* has a spectral gap the series in the definition of σ^2 converges.
- ii. The estimate above is sufficient to apply [2] Theorem 5.1, which yields the desired expansion of the eigenvalue $\chi(t)$ for t near 0.
- iii. The estimate above is sufficient to apply [2] Theorem 5.1, which yields the desired expansion of the eigenvalue $\chi(t)$ for t near 0.
- iv. Similarly we apply [1] Theorem 3.1 to obtain the claimed expansion.

□

8.4 Limit Theorems

Below we collect a technical version of Theorem 1.2

Theorem 8.6. Suppose that $X: [0, 1]^2 \rightarrow \mathbb{R}$ is γ -Hölder for some $\gamma \in (0, 1]$ and $\int_{[0, 1]^2} X d\operatorname{Leb} = 0$.

i. If⁸ $\xi \in L^2$, then as $n \rightarrow \infty$,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} N(0, \sigma^2),$$

⁸See Lemma 8.4

where

$$\sigma^2 := \int_{\Lambda} |\tilde{\xi}|^2 d\lambda + 2 \sum_{k=1}^{\infty} \int_{\Lambda} \tilde{\xi} \circ T^k \tilde{\xi} d\lambda.$$

ii. If $\alpha_0 > \alpha_1$, $\alpha_0 > 1$, and $M_0 > 0$, then as $t \rightarrow 0$

$$\frac{1}{n^{\frac{\alpha_0}{\alpha_0+1}}} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} Z,$$

where

$$E(e^{itZ}) = \exp\left(-A|t|^{1+\frac{1}{\alpha_0}} + iB \operatorname{sgn}(t)|t|^{1+\frac{1}{\alpha_0}}\right),$$

$$A := C_0 \Gamma\left(-\frac{1}{\alpha_0}\right) \cos\left(\frac{1+\frac{1}{\alpha_0}}{2} \pi\right),$$

$$B := C_0 \Gamma\left(-\frac{1}{\alpha_0}\right) \sin\left(\frac{1+\frac{1}{\alpha_0}}{2} \pi\right).$$

iii. If $\alpha_0 = \alpha_1 =: \alpha$, $\alpha > 1$, $M_0 > 0$ and $M_1 < 0$, then as $t \rightarrow 0$,

$$\frac{1}{n^{\frac{\alpha}{\alpha+1}}} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} Z,$$

where

$$E(e^{itZ}) = \exp\left(-A|t|^{1+\frac{1}{\alpha}} + iB \operatorname{sgn}(t)|t|^{1+\frac{1}{\alpha}}\right),$$

$$A := (C_0 + C_1) \Gamma\left(-\frac{1}{\alpha}\right) \cos\left(\frac{1+\frac{1}{\alpha}}{2} \pi\right),$$

$$B := (C_0 - C_1) \Gamma\left(-\frac{1}{\alpha}\right) \sin\left(\frac{1+\frac{1}{\alpha}}{2} \pi\right).$$

iv. Suppose that $\alpha_0 = \alpha_1 = 1$, $M_0 \neq 0$, and $M_1 \neq 0$, then as $t \rightarrow 0$,

$$\frac{1}{\sqrt{n \log(n)}} \sum_{k=0}^{n-1} X \circ B^k \xrightarrow{\text{dist}} N(0, \sigma^2),$$

where

$$\sigma^2 := (C_0 + C_1)$$

Proof. The results follow from arguments similar to those presented in [10] Sections 4.3 and 4.4. For the proof of (iv) it is worth noting that

$$\begin{aligned} \chi\left(\frac{t}{\sqrt{n \log(n)}}\right) &= 1 + (C_0 + C_1) t^2 \frac{1}{n} \left[\frac{\log(t)}{\log(n)} - \frac{\log(\log(n))}{2 \log(n)} - \frac{1}{2} \right] \\ &= 1 - \frac{1}{2} (C_0 + C_1) t^2 \frac{1}{n} [1 - o(1)] \\ &\sim 1 - \frac{1}{2} (C_0 + C_1) t^2 \frac{1}{n}. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \left[\chi \left(t \sqrt{\frac{\log(n)}{n}} \right) \right]^n = \exp \left(-\frac{1}{2} (C_0 + C_1) t^2 \right).$$

□

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